Triangles in graphs without bipartite suspensions

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Abstract

Given graphs T and H, the generalized Turán number $\mathrm{ex}(n, T, H)$ is the maximum number of copies of T in an n-vertex graph with no copies of H. Alon and Shikhelman, using a result of Erdős, determined the asymptotics of $\mathrm{ex}(n, K_3, H)$ when the chromatic number of H is greater than three and proved several results when H is bipartite. We consider this problem when H has chromatic number three. Even this special case for the following relatively simple three chromatic graphs appears to be challenging. The suspension $\widehat{H}$ of a graph H is the graph obtained from H by adding a new vertex adjacent to all vertices of H. We give new upper and lower bounds on $\mathrm{ex}(n, K_3, \widehat{H})$ when H is a path, even cycle, or complete bipartite graph. One of the main tools we use is the triangle removal lemma, but it is unclear if much stronger statements can be proved without using the removal lemma.

1 Introduction

A graph $G = (V(G), E(G))$ consists of a vertex set $V(G)$ and an edge set $E(G) \subseteq \binom{V(G)}{2}$. Let $e(G)$ denote the number of edges of G. Say that G is F-free if G contains no subgraph isomorphic to H. We emphasize that we do not consider induced subgraphs in this definition.

For graphs T and H with no isolated vertices and integer n, the generalized Turán number $\mathrm{ex}(n, T, H)$ is the largest number of copies of T in an H-free n-vertex graph. When $T = K_2$, this is the Turán number $\mathrm{ex}(n, H)$ of the graph H.

The systematic study of $\mathrm{ex}(n, T, H)$ for $T \neq K_2$ was initiated by Alon and Shikhelman [1]. Before then, there had been sporadic results determining this function for several T and H, beginning with $\mathrm{ex}(n, K_t, K_r)$ for $t < r$ (see [2, 6]). Several cases where $H = K_r$ were studied in [14]. There has been a lot of recent activity when $(T, H) = (K_3, C_{2k+1})$ and $(T, H) = (C_5, K_3)$ [1, 3, 9, 13, 12]. In [11], the cases $(T, H) = (P_k, K_{2,t})$ and $(T, H) = (C_k, K_{2,t})$ have also been studied and some generic bounds on $\mathrm{ex}(n, T, H)$ are given. See also [16] for a related result about the number of s-cliques in graphs without cycles of length at least k.

Alon and Shikhelman [1] determine all pairs of graphs T, H with $\mathrm{ex}(n, T, H) = \Theta(n^{|V(T)|})$. Further, they prove that if T and H are trees, then there exists an $m(T, H)$ such that $\mathrm{ex}(n, T, H) = \Theta(n^{m(T,H)})$. They also study the problem when H is a tree and T is a bipartite graph, and give several results on $\mathrm{ex}(n, K_t, H)$ for bipartite H. One general result they prove using a theorem of Erdős [7] is that if the chromatic number $\chi(H) > t$, then

$$\mathrm{ex}(n, K_t, H) = \binom{\chi(H) - 1}{t} \left(\frac{n}{\chi(H) - 1}\right)^t + o(n^t).$$

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In [17], the error term was determined more precisely.

All our results concern $T = K_3$. Since the asymptotic formula of $\mathrm{ex}(n, K_3, H)$ is already known for $\chi(H) > 3$ and [1] studies the case $\chi(H) = 2$ quite extensively, we consider the wide open case $\chi(H) = 3$. Even within this class, we restrict our attention to a very specific and simple family of 3-chromatic graphs. For any graph H, let $\widehat{H}$ denote the suspension $K_1 \vee H$ obtained by adding a new vertex adjacent to every vertex of H. We obtain upper and lower bounds on $\mathrm{ex}(n, K_3, \widehat{H})$ for $H \in \{K_{a,b}, C_{2k}, P_k\}$, where P_k denotes the path on k edges.

Given a graph $G = (V, E)$ and a vertex $v \in V$, let $N_G(v) = \{u \in V : uv \in E\}$ denote the

hood of v in G. For any subset $X \subseteq V$, let $e(X)$ denote the number of edges in the subgraph $G[X]$ induced by X. Let $t(G)$ denote the number of triangles in G.

If G is $\widehat{H}$-free, then $N_G(v)$ is H-free for every $v \in V$. This implies that

$$t(G) = \frac{1}{3}\sum_{v \in V} e(N_G(v)) \leq \frac{1}{3}\sum_{v \in V} \mathrm{ex}(|N_G(v)|, H) \leq \frac{n}{3} \cdot \mathrm{ex}(n, H).$$

Hence,

$$\mathrm{ex}(n, K_3, \widehat{H}) \leq \frac{n}{3} \cdot \mathrm{ex}(n, H). \tag{1.1}$$

All our results give improvements on (1.1). For our first result $H = K_{a,b}$ and $\widehat{H} = K_{1,a,b}$, where $1 \leq a \leq b$. Here (1.1) combined with the Kövari-Sós-Turán theorem [15], which asserts that $\mathrm{ex}(n, K_{a,b}) = O(n^{2-\frac{1}{a}})$ yields $\mathrm{ex}(n, K_3, K_{1,a,b}) = O(n^{3-\frac{1}{a}})$. We improve this as follows.

Theorem 1.1. *For fixed $1 \leq a \leq b$ and $n \to \infty$,*

$$\mathrm{ex}(n, K_3, K_{1,a,b}) = o(n^{3-\frac{1}{a}}). \tag{1.2}$$

Notice that setting $a = b = 2$ in (1.2) yields $\mathrm{ex}(n, K_3, K_{1,2,2}) = o(n^{5/2})$, where $K_{1,2,2} = \widehat{C}_4$. This is related to a question in [18], where the authors asked whether $\mathrm{ex}(n, K_3, K_{1,2,2}) = O(n^2)$. While this remains open we do give a quadratic lower bound in Proposition 3.1. Narrowing the (huge) gap in the bounds

$$\Omega(n^2) < \mathrm{ex}(n, K_3, K_{1,2,2}) < o(n^{5/2})$$

is perhaps the most basic and attractive open problem raised in this paper.

Our next result concerns $H = C_{2k}$. Here (1.1) together with the classical bound $\mathrm{ex}(n, C_{2k}) = O(n^{1+\frac{1}{k}})$ of Bondy-Simonovits [4] yields $\mathrm{ex}(n, K_3, \widehat{C}_{2k}) = O(n^{2+\frac{1}{k}})$.

Theorem 1.2. *For fixed $k \geq 2$ and $n \to \infty$,*

$$ex(n, K_3, \widehat{C}_{2k}) = o(n^{2+\frac{1}{k}}). \tag{1.3}$$

The same lower bound construction for $\mathrm{ex}(n, K_3, K_{1,2,2})$ yields $\mathrm{ex}(n, K_3, \widehat{C}_{2k}) = \Omega(n^2)$.

Our final results concern $\mathrm{ex}(n, K_3, \widehat{P}_k)$ for $k \geq 3$. We begin with a simple proposition.

Proposition 1.3. *Let $n \geq k \geq 3$. Then*

$$\left\lfloor \frac{k-1}{2} \right\rfloor \cdot \frac{n^2}{8} \leq ex(n, K_3, \widehat{P}_k) \leq \frac{k-1}{12} \cdot n^2 + \frac{(k-1)^2}{12} \cdot n, \tag{1.4}$$

where the lower bound holds when n is a multiple of $4\lfloor \frac{k-1}{2} \rfloor$.

We believe that the lower bound above is asymptotically tight for all fixed $k \geq 3$ and prove this for the first three cases $k = 3$, 4 and 5.

Theorem 1.4. *For $k = 3$, 4 and 5,*

$$ex(n, K_3, \widehat{P}_k) = \left\lfloor \frac{k-1}{2} \right\rfloor \cdot \frac{n^2}{8} + o(n^2). \tag{1.5}$$

When $k = 3$ or $k = 5$, the error term can be improved to $O(n)$.

In Section 2, we present some preliminary results that we use in our proofs. In Section 3 we prove Theorem 1.1, in Section 4 we prove Theorem 1.2 and in Section 5 we prove Proposition 1.3 and Theorem 1.4. Finally, we present some concluding remarks in Section 6.

Note. After making this paper public, we learnt that Theorem 1.1 has also been proved by several other researchers and Theorem 1.2 has been proved by Methuku, Grósz, Tompkins (using a different proof). These works are unpublished.

2 Preliminaries

In this section, we describe some preliminary tools that will be used in our proofs. The most important tool for proving Theorems 1.1 and 1.2 is the triangle removal lemma [5, 10, 21]. The specific form that we shall be using appears as Theorem 2.1 in [5].

Lemma 2.1 (Ruzsa-Szemerédi [21])**.** *Suppose $\epsilon > 0$. Let $\delta = \delta(\epsilon)$ be such that $\frac{1}{\delta}$ is a tower of twos of height $684\log(\frac{1}{\epsilon})$. If G is a graph on n vertices with at least ϵn^2 edge-disjoint triangles, then G contains at least δn^3 triangles.*

Recall that $t(G)$ is the number of triangles in G.

Lemma 2.2 (Nordhaus-Stewart [19])**.** *For any graph G on n vertices,*

$$t(G) \geq \frac{e(G)}{3n} \cdot \left(4e(G) - n^2\right).$$

A k-uniform hypergraph $H = (V(H), E(H))$, consists of a vertex set $V(H)$ and edge set $E(H) \subseteq \binom{V(H)}{k}$. We write $e(H) = |E(H)|$. A subset $X \subseteq V(H)$ of vertices is an independent set if $e \not\subseteq X$ for any $e \in E(H)$. The independence number of H, denoted by $\alpha(H)$, is the largest size of an independent set in H.

Lemma 2.3 (Spencer [22])**.** *Let $H = (V, E)$ be an r-uniform hypergraph on n vertices, and let $d(H) = \frac{re(H)}{n}$ denote the average degree of H. Then,*

$$\alpha(H) \geq \frac{r-1}{r} \cdot \left(\frac{n}{d(H)^{\frac{1}{r-1}}}\right).$$

Finally, we require a result that is a direct consequence of the proof of the Erdős-Gallai theorem [8] for cycles, which states that every graph with average degree at least k contains a cycle of length at least $k+1$. Recall that a chord in a cycle is an edge between any two non-adjacent vertices of the cycle.

Lemma 2.4 (Erdős-Gallai [8])**.** *Let $k \geq 3$ be an integer. If G is a graph of average degree at least k, then G contains a cycle of length at least $k+1$ with a chord. In particular, G also contains a path of length at least k.* $\square$

3 Suspension of complete bipartite graphs

Our goal in this short section is to prove Theorem 1.1 and give a lower bound on $\mathrm{ex}(n, K_3, K_{1,2,2})$ via Proposition 3.1. Given a graph G and $X \subseteq V(G)$, let $N_G(X) = \bigcap_{v\in X} N_G(v)$ denote the common neighborhood of all vertices from X.

Proof of Theorem 1.1. Recall that we are to show that for $1 \le a \le b$, $\mathrm{ex}(n, K_3, K_{1,a,b}) = o(n^{3-\frac{1}{a}})$. Let n be sufficiently large, and G be an n-vertex graph which is $K_{1,a,b}$-free. By the Kövari-Sós-Turán theorem [15] and (1.1), we know that there exists $c = c_b$ such that

$$t(G) < c_b n^{3-\frac{1}{a}}. \tag{3.1}$$

Now, suppose $\epsilon > 0$ is fixed. Assume that n is sufficiently large, and that G is a $K_{1,a,b}$-free graph with $t(G) \ge \epsilon n^{3-\frac{1}{a}}$. Construct an a-uniform hypergraph H whose vertices are the triangles of G, and let $\{T_1, \dots, T_a\}$ be an edge of H if the triangles $T_1, \dots, T_a$ all share a common edge in G. Then

$$e(H) = \sum_{\{v_1,\dots,v_a\}\in\binom{V(G)}{a}} e(N_G(\{v_1, \dots, v_a\})).$$

As G is $K_{1,a,b}$-free, $N_G(\{v_1, \dots, v_a\})$ has no vertex of degree at least b. This implies that

$$e(N_G\{v_1, \dots, v_a\}) < \frac{b}{2} \cdot |N_G(\{v_1, \dots, v_a\})|.$$

Consequently,

$$e(H) < \sum_{\{v_1,\dots,v_a\}\in\binom{[n]}{a}} \frac{b}{2} \cdot |N_G(v_1, \dots, v_a)| = \frac{b}{2} \sum_{v\in V(G)} \binom{\deg(v)}{a} < b \sum_{v\in V(G)} \deg(v)^a < b \cdot n^{a+1}.$$

This implies that $d(H) < \frac{ab\cdot n^{a+1}}{t(G)}$. Using Lemma 2.3,

$$\alpha(H) > \frac{a-1}{a} \cdot \frac{t(G)}{\left(\frac{ab\cdot n^{a+1}}{t(G)}\right)^{\frac{1}{a-1}}} = c \cdot t(G)^{1+\frac{1}{a-1}} \cdot n^{-\frac{a+1}{a-1}},$$

where $c = (a-1)/(a(ab)^{1/(a-1)})$. Recalling that $t(G) \ge \epsilon n^{3-\frac{1}{a}}$ and letting $\epsilon' = c \cdot \epsilon^{1+\frac{1}{a-1}}$, we obtain

$$\alpha(H) > \epsilon' n^{\frac{a}{a-1}\cdot\frac{3a-1}{a}} \cdot n^{-\frac{a+1}{a-1}} = \epsilon' n^2.$$

Let I be a maximum independent set of H. Create an auxiliary graph H' with vertex set I, and join two vertices of H' iff the triangles corresponding to them share an edge. Every triangle from I can be adjacent to at most $3(a-1)$ other triangles from I. Therefore, $\deg_{H'}(i) < 3a$ for every $i \in I$, and hence by Lemma 2.3 H' has an independent set of size at least $\frac{|I|}{6a} > \frac{\epsilon' n^2}{6a}$. The triangles corresponding to this independent set are edge-disjoint. Therefore $t(G) \ge \delta n^3$ where $\delta = \delta(\frac{\epsilon'}{6a})$ is obtained from Lemma 2.1. However $t(G) < c_b n^{3-\frac{1}{a}}$ by (3.1) and this implies that $\delta n^3 \le \frac{c_b}{3} \cdot n^{3-\frac{1}{a}}$, a contradiction for sufficiently large n. $\square$

Plugging in $a = b = 2$ in Theorem 1.1, we get the bound $\mathrm{ex}(n, K_3, K_{1,2,2}) = o(n^{5/2})$. We now describe a lower bound construction for $\mathrm{ex}(n, K_3, K_{1,2,2})$.

Proposition 3.1. *When n is a multiple of 4,*

$$ex(n, K_3, K_{1,2,2}) \ge \frac{n^2}{4}.$$

Proof of Proposition 3.1. Let $H_n = (A, B)$ be the complete bipartite graph with $|A| = |B| = \frac{n}{2}$, with additional edges in both the parts such that $H_n[A]$ and $H_n[B]$ are matchings of size $\frac{n}{4}$. Observe that the neighborhood of every vertex in H_n consists of $\frac{n}{4}$ edge-disjoint triangles sharing a common vertex, and hence is C_4-free. Thus H_n is $K_{1,2,2}$-free. On the other hand, every triangle of H_n either has an edge inside A or an edge inside B, implying

$$t(H_n) = e(A) \cdot |B| + e(B) \cdot |A| = 2 \cdot \frac{n}{4} \cdot \frac{n}{2} = \frac{n^2}{4}.$$

This proves that whenever $4 \mid n$, $\mathrm{ex}(n, K_3, K_{1,2,2}) \geq n^2/4$. $\square$

4 Suspension of even cycles

Our goal in this section is to prove Theorem 1.2. Before proceeding with the proof, we prove Lemma 4.2 which gives an upper bound on the number of paths of length k in a C_{2k}-free graph. The main idea behind the lemma is the technique used in [20, 23, 24] to prove upper bounds on $\mathrm{ex}(n, C_{2k})$ by analyzing the breadth-first search tree from any vertex.

Given a graph G and a vertex $r \in V(G)$, a breadth-first search tree T of G rooted at r is constructed as follows. Let $L_0 = \{r\}$. For $i \geq 1$, let $L_i \subseteq V(G)$ be the set of all vertices in $V(G)$ which are at distance i from vertex r. The vertex subset L_i is called the i'th level of T. The tree T consists of vertex set $V(G)$ and edge set a subset of the edges of G between levels L_i and L_{i+1}, $i \geq 0$, such that every vertex in L_{i+1} is incident to exactly one vertex in L_i.

For $i \geq 0$, let $G[L_i]$ be the subgraph of G induced by L_i, and let $G[L_i, L_{i+1}]$ be the bipartite subgraph of G with parts (L_i, L_{i+1}) and edges exactly those of G with one endpoint in L_i and another in L_{i+1}.

We now quote Lemma 3.5 from [24] in the form that we shall be using.

Lemma 4.1 (Verstraëte [24])**.** *Let T be a breadth-first search tree in a graph G, with levels $L_0, L_1, \ldots$. Let Θ_ℓ denote the set of cycles of length ℓ with a chord. Then,*

(a) If $G[L_i]$ contains an element of Θ_ℓ, then there exists $m \leq i$ such that G contains cycles C_{2m+1} ,C_{2m+2}, $\ldots$, $C_{2m+\ell-1}$.

(b) If $G[L_i, L_{i+1}]$ contains an element of Θ_ℓ, then there exists $m \leq i$ such that G contains cycles C_{2m+2}, C_{2m+4}, $\ldots$, $C_{2m+\ell'}$, where ℓ' is the largest even integer less than ℓ.

Let $p_k(G)$ denote the number of paths of length k in a graph G. Here each subgraph isomorphic to P_k is counted twice, once for each ordering of its vertices along the path.

Lemma 4.2. *Let $k \geq 2$ be an integer, $0 < \epsilon < 1$ and $n > (20k/\epsilon)^k$. Let F be a C_{2k}-free graph on n vertices with minimum degree at least $\epsilon n^{1/k}$. Then*

$$p_k(F) \leq \left(\frac{2k}{\epsilon}\right)^{(k-1)k} n^2.$$

Proof of Lemma 4.2. We first prove that F has bounded maximum degree. Suppose for contradiction that there exists $v \in V(F)$ with

$$\deg(v) \geq \left(\frac{2k}{\epsilon}\right)^{k-1} \cdot n^{1/k}.$$

Consider the breadth-first search tree of F starting at v. For $i \geq 0$, let L_i be the ith level of this breadth-first search tree. By assumption, $|L_1| \geq \left(\frac{2k}{\epsilon}\right)^{k-1} \cdot n^{1/k}$. Denote by $e(L_i, L_{i+1})$ the number of edges in $G[L_i, L_{i+1}]$. Let us prove that for every $1 \leq i < k$,

$$e(L_i) \leq (k-1)|L_i| \quad \text{and} \quad e(L_i, L_{i+1}) \leq (k-1)(|L_i| + |L_{i+1}|). \tag{4.1}$$

Indeed, if $e(L_i) > (k-1)|L_i|$, then by Lemma 2.4, $F[L_i]$ contains a cycle of length ℓ with a chord, where $\ell \geq 2k-1$. Now, we apply Lemma 4.1 (a) to obtain an integer $m \leq i$ such that F contains cycles of lengths $2m+1, 2m+2, \ldots, 2m+\ell-1$. Since $\ell \geq 2k-1$ and $1 \leq m < k$, we have $2m+1 \leq 2k \leq 2m+\ell-1$. Then F contains a C_{2k}, contradiction. Similarly, if $e(L_i, L_{i+1}) > (k-1)(|L_i| + |L_{i+1}|)$, then Lemma 2.4 gives us a cycle of length ℓ with a chord in $F[L_i, L_{i+1}]$ where $\ell \geq 2k-1$. Then Lemma 4.1 (b) gives an integer $m \leq i$ such that F contains cycles of lengths $2m+2, 2m+4, \ldots, 2m+\ell$. As $\ell \geq 2k-1$ and $1 \leq m < k$, we have $2m+2 \leq 2k \leq 2m+2k-1$. This implies that F contains a C_{2k}, again a contradiction.

Claim 4.3. For every $i \geq 0$,

$$|L_{i+1}| \geq \frac{\epsilon n^{1/k}}{2k} \cdot |L_i|. \tag{4.2}$$

Proof of Claim 4.3. We use induction on i. Note that $|L_0| = 1$ and

$$|L_1| = \deg(v) \geq \left(\frac{2k}{\epsilon}\right)^{k-1} \cdot n^{1/k} > \frac{\epsilon n^{1/k}}{2k} = \frac{\epsilon n^{1/k}}{2k}|L_0|.$$

Moreover, for $i \geq 1$ and any vertex $u \in L_i$, $N_G(u) \subseteq L_{i-1} \cup L_i \cup L_{i+1}$. Thus, (4.1) implies

$$\begin{aligned}
k(|L_i| + |L_{i+1}|) + 2k|L_i| + k(|L_i| + |L_{i-1}|) &> e(L_i, L_{i+1}) + 2e(L_i) + e(L_i, L_{i-1}) \\
&= \sum_{u \in L_i} \deg(u) \\
&\geq \epsilon n^{1/k} \cdot |L_i|.
\end{aligned}$$

Consequently,

$$|L_{i+1}| > \left(\frac{\epsilon n^{1/k}}{k} - 4\right) \cdot |L_i| - |L_{i-1}|. \tag{4.3}$$

By the induction hypothesis we may assume that $|L_{i-1}| \leq \frac{2k}{\epsilon n^{1/k}} \cdot |L_i|$. Thus, (4.3) implies

$$|L_{i+1}| > \left(\frac{\epsilon n^{1/k}}{k} - 4 - \frac{2k}{\epsilon n^{1/k}}\right) \cdot |L_i| > \frac{\epsilon n^{1/k}}{2k} \cdot |L_i|$$

since $n > (20k/\epsilon)^k$. This finishes the proof of Claim 4.3.

Now, by applying Claim 4.3 iteratively, we obtain

$$|L_k| \geq \left(\frac{\epsilon n^{1/k}}{2k}\right)^{k-1} \cdot |L_1| \geq \left(\frac{\epsilon n^{1/k}}{2k}\right)^{k-1} \cdot \left(\frac{2k}{\epsilon}\right)^{k-1} \cdot n^{1/k} = n,$$

a contradiction. Thus, $\deg(u) \leq \left(\frac{2k}{\epsilon}\right)^{k-1} \cdot n^{1/k}$ for every $u \in V(F)$. Therefore, if $\Delta(F)$ is the maximum degree of F,

$$p_k(F) \leq n \cdot \Delta(F)^k \leq \left(\frac{2k}{\epsilon}\right)^{k(k-1)} \cdot n^2,$$

as desired. □

For a graph G and edge $uv \in E(G)$, the codegree of uv is $\deg_G(u,v) = |N_G(\{u,v\})|$.

Proof of Theorem 1.2. Fix $\epsilon > 0$ and let n be sufficiently large. Suppose $G = (V, E)$ is a graph on n vertices satisfying $t(G) \geq \epsilon n^{2+\frac{1}{k}}$. We wish to show that G contains a copy of $\widehat{C}_{2k}$. Suppose, on the contrary, that G is $\widehat{C}_{2k}$-free. First, we iteratively delete edges of G with codegree less than $\frac{\epsilon n^{1/k}}{10}$ in the current graph, until there are no such edges left. Since we delete fewer than $e(G) \cdot \frac{\epsilon n^{1/k}}{10}$ triangles, we are left with a graph G' satisfying

$$t(G') > t(G) - e(G) \cdot \frac{\epsilon n^{1/k}}{10} > \epsilon n^{2+\frac{1}{k}} - \frac{\epsilon n^{2+\frac{1}{k}}}{10} \geq \frac{9\epsilon}{10} \cdot n^{2+\frac{1}{k}},$$

and $\deg_{G'}(u,v) \ge \frac{\epsilon n^{1/k}}{10}$ for every $uv \in E(G')$.

Next, create an auxiliary graph H whose vertices are the triangles of G', and two vertices of H are adjacent iff their corresponding triangles share a common edge. By Lemma 2.3, $\alpha(H) > \frac{t(G')}{2d(H)}$. Let

$$\gamma := \frac{(\epsilon/20)^{k^2}}{k^{(k-1)k}} > 0.$$

If $\frac{t(G')}{2d(H)} > \frac{\gamma}{2}n^2$, then this gives us $\frac{\gamma}{2}n^2$ edge-disjoint triangles in G, and this implies that $t(G) > \delta n^3$ where $\delta = \delta(\frac{\gamma}{2})$ from Lemma 2.1. However, we also have $t(G) < c_k n^{2+\frac{1}{k}}$ for some constant c_k by (1.1) and this is a contradiction since n is sufficiently large. Therefore, we may assume $\frac{t(G')}{d(H)} \le \gamma n^2$ and this implies

$$d(H) \ge \frac{t(G')}{\gamma n^2} \ge \frac{9\epsilon}{10\gamma} \cdot n^{1/k}$$

and hence

$$e(H) \ge \frac{9\epsilon n^{1/k}}{20\gamma} \cdot t(G') \ge \frac{81\epsilon}{200\gamma} \cdot n^{2+\frac{2}{k}}.$$

Let us now bound X, the number of copies of $\widehat{P}_k$ in G' in two different ways. For every $v \in V(G')$, let G'_v denote the subgraph of G' induced by $N_{G'}(v)$. Let $\delta(F)$ denote the minimum degree of F for any graph F. By the assumption on the minimum codegree of edges in G', $\delta(G'_v) \ge \frac{\epsilon n^{1/k}}{10}$. Hence applying Lemma 4.2 with ϵ replaced by $\frac{\epsilon}{10}$,

$$\begin{aligned} X &\le \sum_{v \in V(G')} p_k(G'_v) \le n \cdot \left(\frac{20k}{\epsilon}\right)^{k(k-1)} \cdot n^2 \\ &= \left(\frac{20k}{\epsilon}\right)^{k(k-1)} \cdot n^3. \end{aligned} \tag{4.4}$$

On the other hand, we can first fix two adjacent triangles in G' and then keep growing it to a $\widehat{P}_k$ by using the minimum codegree condition of G'. Since $\delta(H) \ge \frac{\epsilon n^{1/k}}{10}$, this implies that for large enough n,

$$\begin{aligned} X &\ge \frac{1}{2}e(H) \cdot (\delta(H)-2) \cdot (\delta(H)-3) \cdots (\delta(H)-k+1) \\ &\ge \frac{1}{2}e(H) \cdot (\delta(H)-k)^{k-2} \\ &\ge \frac{81\epsilon}{400\gamma} \cdot n^{2+\frac{2}{k}} \cdot \left(\frac{\epsilon n^{1/k}}{20}\right)^{k-2} \\ &= \frac{81\epsilon^{k-1}}{20^k \cdot \gamma} \cdot n^3. \end{aligned} \tag{4.5}$$

The factor $\frac{1}{2}$ in (4.5) is to balance out over-counting the same $\widehat{P}_k$ from its two ends. Comparing (4.4) and (4.5), we obtain

$$\left(\frac{20k}{\epsilon}\right)^{k(k-1)} \ge \frac{81\epsilon^{k-1}}{20^k \cdot \gamma},$$

implying

$$\gamma \ge \frac{81\epsilon^{k^2}}{20^{k^2} \cdot k^{k(k-1)}} = 81 \cdot \frac{(\epsilon/20)^{k^2}}{k^{k(k-1)}} = 81\gamma,$$

a contradiction. This completes the proof of Theorem 1.2. □

5 Suspension of paths

In this section, we prove Proposition 1.3 and Theorem 1.4.

5.1 Proof of Proposition 1.3

First, we show the upper bound in (1.4). Let $k \geq 3$ be fixed, and let G be a graph on n vertices which is $\widehat{P}_k$-free. We need to show that $t(G) \leq \frac{(k-1)}{12} \cdot n^2 + \frac{(k-1)^2}{12} \cdot n$.

Note that the neighborhood of every vertex $v \in V(G)$ is P_k-free. Thus by Lemma 2.4, the average degree of the subgraph of G induced by $N_G(v)$ is at most $k-1$. Hence,

$$e(N_G(v)) \leq \frac{k-1}{2} \cdot \deg_G(v).$$

Summing up this inequality over all vertices $v \in V(G)$,

$$3t(G) = \sum_{v \in V(G)} e(N_G(v)) \leq \frac{k-1}{2} \cdot 2e(G) = (k-1)e(G),$$

giving us

$$t(G) \leq \frac{k-1}{3} \cdot e(G). \tag{5.1}$$

This, in conjunction with Lemma 2.2, gives us

$$\frac{k-1}{3} \cdot e(G) \geq t(G) \geq \frac{e(G)}{3n} \cdot (4e(G) - n^2),$$

which simplifies to

$$e(G) \leq \frac{n^2}{4} + \frac{(k-1)n}{4}.$$

The conclusion of the upper bound follows from plugging this inequality back into (5.1).

Now we prove the lower bound in (1.4). Let n be a multiple of $4\left\lfloor \frac{k-1}{2} \right\rfloor$. We shall construct a $\widehat{P}_k$-free graph $F_{n,k}$ on n vertices with $t(F_{n,k}) \geq \left\lfloor \frac{k-1}{2} \right\rfloor \cdot \frac{n^2}{8}$.

Let $F_{n,k} = (A, B)$ be the complete bipartite graph with parts A, B with $|A| = |B| = \frac{n}{2}$, with additional edges in A such that $F_{n,k}[A]$ is a disjoint union of $K_{\left\lfloor \frac{k-1}{2} \right\rfloor, \left\lfloor \frac{k-1}{2} \right\rfloor}$. Then,

$$e(A) = \left\lfloor \tfrac{k-1}{2} \right\rfloor^2 \cdot \frac{n}{4\left\lfloor \frac{k-1}{2} \right\rfloor} = \left\lfloor \tfrac{k-1}{2} \right\rfloor \cdot \tfrac{n}{4}.$$

Every triangle of $F_{n,k}$ consists of an edge from $F_{n,k}[A]$ and a vertex from B. Hence,

$$t(F_{n,k}) = \left\lfloor \tfrac{k-1}{2} \right\rfloor \cdot \tfrac{n}{4} \cdot \tfrac{n}{2} = \left\lfloor \tfrac{k-1}{2} \right\rfloor \cdot \tfrac{n^2}{8}.$$

Further, $F_{n,k}$ is $\widehat{P}_k$-free, since the neighborhood of every vertex in B is a disjoint union of $K_{\left\lfloor \frac{k-1}{2} \right\rfloor, \left\lfloor \frac{k-1}{2} \right\rfloor}$, and the neighborhood of every vertex in A is isomorphic to $K_{\left\lfloor \frac{k-1}{2} \right\rfloor, \frac{n}{2}}$. □

5.2 Proof of Theorem 1.4

We will use some ideas from [9], and define the concepts of *triangle-connectivity* and *blocks*. In what follows, a triangle T in a graph G is a set of three edges $\{ab, bc, ca\}$ that form a K_3 in G. Subsequently, we shall denote such a triangle simply as *abc*.

Definition 5.1 (Triangle-connectivity)**.** Given a graph G and two distinct edges $e, e' \in E(G)$, say that e and e' are *triangle-connected* if there is a sequence of triangles $\{T_1, \ldots, T_k\}$ of G, such that $e \in T_1$, $e' \in T_k$, and T_i and T_{i+1} share a common edge for every $1 \leq i \leq k-1$. A subgraph $H \subseteq G$ is *triangle-connected* if e and e' are triangle-connected for every two distinct $e, e' \in E(H)$.

It is straightforward to check that triangle-connectivity is an equivalence relation on $E(G)$ (assuming reflexivity as part of the definition).

Definition 5.2 (Triangle block)**.** A *triangle block*, or simply a *block* in a graph G is a subgraph H whose edges form an equivalence class of the *triangle-connectivity* relation on $E(G)$.

In other words, a subgraph $H \subseteq G$ is a triangle block if it is edge-maximally triangle-connected. By definition, the triangle blocks of a graph G are edge-disjoint.

5.2.1 Proof of Theorem 1.4 for $\mathbf{k = 3}$.

Suppose G is a graph on n vertices which is $\widehat{P}_3$-free. We will prove using induction on n, that

$$t(G) < \frac{n^2}{8} + 3n. \tag{5.2}$$

This inequality is true for $n = 3$ as $t(G) \leq 1$ for any graph G on 3 vertices. Now, fix an $n > 3$ and a graph G on n vertices which is $\widehat{P}_3$-free. Assume that (5.2) holds for $\widehat{P}_3$-free graphs on less than n vertices.

We may assume without loss of generality that every edge of G lies in a triangle, otherwise we may delete it from G without changing $t(G)$. For a vertex $v \in V(G)$, let $t(v) = t_G(v)$ denote the number of triangles in G containing v. By definition, $t_G(v) = e(N_G(v))$.

We first prove that if G has a copy of K_4, then $t(G) < \frac{n^2}{8} + 3n$, hence completing the induction step.

Suppose G has a copy of K_4 with vertices labeled a_1, a_2, a_3, a_4. Let $X = V(G) \setminus \{a_1, a_2, a_3, a_4\}$, and $A_i = N_G(a_i) \cap X$, $i = 1, \dots, 4$. If $x \in A_1 \cap A_2$, then we can find a $\widehat{P}_3$ formed by a_1, x, a_2, a_3, a_4 in the neighborhood of a_1. Thus, $A_1 \cap A_2 = \varnothing$, and by symmetry the A_i's are mutually disjoint. Hence, $|A_1|+|A_2|+|A_3|+|A_4| \leq |X| = n-4$. This implies that one of the A_i's has size $\leq \frac{n-4}{4}$. Using Lemma 2.4 in the neighborhood of a_i,

$$t(a_i) = 3 + e(A_i) \leq 3 + |A_i| \leq 3 + \frac{n-4}{4} = \frac{n+8}{4}.$$

Now let $G' = G - a_i$. As G was $\widehat{P}_3$-free, so is G'. Hence by the induction hypothesis,

$$t(G') < \frac{(n-1)^2}{8} + 3(n-1).$$

This implies,

$$t(G) = t(G') + t(a_i) < \frac{(n-1)^2}{8} + 3(n-1) + \frac{n+8}{4} < \frac{n^2}{8} + 3n,$$

as desired. We may now assume that G is K_4-free.

Let B_s denote the *book graph* on $s+2$ vertices, consisting of s triangles all sharing a common edge.

Claim 5.3. Every triangle block of G is isomorphic to B_s for some $s \geq 1$.

Proof of Claim 5.3. Let $H \subseteq G$ be an arbitrary triangle block. If H contains only one or two triangles, it is isomorphic to B_1 or B_2. Suppose H contains at least three triangles. Let two of them be abx_1 and abx_2 (Figure 5.1). If another triangle is of the form ax_1y for some $y \in V(H)$, then there are two possible cases. If $y \neq x_2$, then $N_H(a)$ contains the 3-path x_2bx_1y. Otherwise, if $y = x_2$, then the vertices a, b, x_1, x_2 create a K_4. Similarly, no triangle contains any of the edges bx_1, ax_2, bx_2. Therefore all triangles in H contain ab and $H \cong B_s$ for some $s \geq 1$.

Claim 5.3 implies that G comprises r edge-disjoint blocks isomorphic to books for some $r \geq 1$. Let the blocks of G be isomorphic to $B_{s_1}, \dots, B_{s_r}$, where $s_1, \dots, s_r \geq 1$. Then,

$$t(G) = s_1 + \cdots + s_r \quad \text{and} \quad e(G) = 2(s_1 + \cdots + s_r) + r = 2t(G) + r.$$

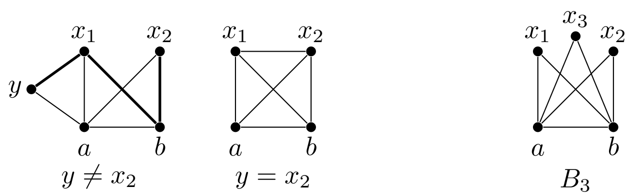

Figure 5.1: (left): third triangle on ax_1, (right): third triangle on ab

Hence, $t(G) < e(G)/2$. Finally, we apply Lemma 2.2 on G to obtain

$$\frac{e(G)}{2} > t(G) \geq \frac{e(G)}{3n} \cdot (4e(G) - n^2),$$

implying

$$e(G) < \frac{n^2}{4} + \frac{3n}{8}.$$

Therefore $t(G) < \frac{n^2}{8} + \frac{3n}{16} < \frac{n^2}{8} + 3n$, completing the induction step. □

5.2.2 Proof of Theorem 1.4 for k = 4.

Suppose $\epsilon > 0$ and n is sufficiently large. Let G be any graph on n vertices which is $\widehat{P}_4$-free, such that

$$t(G) \geq \frac{n^2}{8} + 13\epsilon n^2.$$

The very first step of the proof is to remove copies of K_4 and $K_{1,2,2}$ from G while still maintaining $t(G) \geq \frac{n^2}{8} + \epsilon n^2$. We achieve this by means of the triangle removal lemma. First, we make an observation which follows immediately from Lemma 2.1, (1.1) and Lemma 2.4 for large n.

$$G \text{ cannot have } \epsilon n^2 \text{ edge-disjoint triangles.} \tag{5.3}$$

Without loss of generality, we may assume that every edge of G is contained in a triangle. We shall use (5.3) to remove all copies of the following six graphs in this order: K_5; K_5^-; K_4; $K_{2,2,2}$; $Q_{3,2} = \overline{K_2} \vee P_3$; and $K_{1,2,2}$ (Figure 5.2).

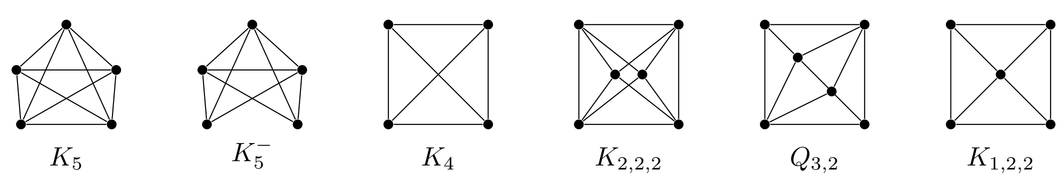

Figure 5.2: The graphs K_5, K_5^-, K_4, $K_{2,2,2}$, $Q_{3,2}$ and $K_{1,2,2}$.

- **Step 1:** Cleaning K_5's.

  If G contains a K_5 with vertices a_1, a_2, a_3, a_4, a_5, then it has to be a block by itself. This is because if there is a vertex $x \neq a_i$ with $xa_1, xa_2 \in E(G)$, then $N_G(a_1)$ contains the path $a_5a_4a_3a_2x$, contradiction. Hence, all the K_5's in G are edge-disjoint.

  If G has more than ϵn^2 copies of K_5, then by taking one triangle from each K_5, we get ϵn^2 edge-disjoint triangles in G, contradicting (5.3). Therefore, G has at most ϵn^2 copies of K_5.

  We now delete one edge from each copy of K_5 in G, and lose at most $3\epsilon n^2$ triangles from G. So, we may assume $t(G) \geq \frac{n^2}{8} + 10\epsilon n^2$, and G is $\{\widehat{P}_4, K_5\}$-free.

- **Step 2:** Cleaning K_5^-'s.

  Suppose G contains a K_5^-. Observe that if we have a new vertex $x \neq a_i$ which is adjacent to two endpoints of any edge of this K_5^-, it would create a copy of $\widehat{P}_4$ (see Figure 5.3 (left)). Thus, the only way two K_5^-'s can intersect in an edge is if they share the same five vertices. This would give us a K_5 in G, a contradiction. Therefore, the copies of K_5^- are all edge-disjoint.

  Hence, if G has more than ϵn^2 copies of K_5^-, we again obtain at least ϵn^2 edge-disjoint triangles in G, contradicting (5.3). So G has at most ϵn^2 copies of K_5^-.

  Deleting one edge from each copy of K_5^- in G, we lose at most $3\epsilon n^2$ triangles in the process. After deletion, we still have $t(G) \geq \frac{n^2}{8} + 7\epsilon n^2$, and we can further assume that G is $\{\widehat{P}_4, K_5^-\}$-free.

Figure 5.3: (left): K_5^-'s are edge-disjoint; (right): K_4's are edge-disjoint.

- **Step 3:** Cleaning K_4's.

  First, we claim that any two copies of K_4 in G are edge-disjoint. If not, then they can only intersect in one edge, or three edges. If they intersect in one edge, we find a $\widehat{P}_4$, and otherwise we get a K_5^- in G (see Figure 5.3 (right); the intersecting edges are illustrated in bold). Hence, all K_4's in G are edge-disjoint.

  Consequently, if there are more than ϵn^2 copies of K_4 in G, taking one triangle from each copy gives us ϵn^2 edge-disjoint triangles, contradicting (5.3) again.

  Now, observe that for every K_4 in G with vertices $\{a, b, c, d\}$, either the edge ab or the edge bc has codegree exactly 2. Otherwise, let $x, y \in V(G)$ be such that xab and ybc are triangles in G. If $x = y$, then $xabcd$ is a K_5^-, and otherwise $xadcy$ is a P_4 in the neighborhood of b. Hence, whenever G contains a K_4, we can remove an edge of codegree 2 from it. Then G loses at most $2\epsilon n^2$ triangles. Thus, we assume that $t(G) \geq \frac{n^2}{8} + 5\epsilon n^2$, and that G is $\{\widehat{P}_4, K_4\}$-free.

- **Step 4:** Cleaning $K_{2,2,2}$'s.

  By assumption, G contains no copy of $\widehat{P}_4$ and K_4. Fix a $K_{2,2,2}$ of G with vertices c_1, c_2 in the center and a_1, a_2, a_3, a_4 forming the outer C_4. Let $X = V(G) \setminus \{c_1, c_2, a_1, a_2, a_3, a_4\}$. Denote $C_i = N_G(c_i) \cap X$ for $i = 1, 2$, and $A_i = N_G(a_i) \cap X$ for $i = 1, \ldots, 4$. Since G is $\widehat{P}_4$-free, we deduce that $A_i \cap A_{i+1} = \varnothing$ and $A_i \cap C_j = \varnothing$ for every i, j (here we denote $A_5 := A_1$). This is shown in Figure 5.4, by assuming $x \in A_1 \cap A_2$ and then $x \in A_1 \cap C_2$, and finding copies of $\widehat{P}_4$ in either case. This implies that the $K_{2,2,2}$'s are themselves triangle blocks of G, hence they are mutually edge-disjoint.

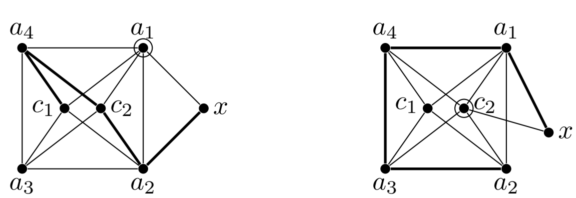

Figure 5.4: $K_{2,2,2}$ is a block by itself.

So, if G has at least ϵn^2 copies of $K_{2,2,2}$, then by taking one triangle from each $K_{2,2,2}$ we obtain at least ϵn^2 edge-disjoint triangles in G, contradicting (5.3).

Deleting one edge from each $K_{2,2,2}$, we lose at most $2\epsilon n^2$ triangles from G. Thus, we may assume that $t(G) \geq \frac{n^2}{8} + 3\epsilon n^2$, and that G is $\{\widehat{P}_4, K_4, K_{2,2,2}\}$-free.

- **Step 5:** Cleaning $Q_{3,2}$'s.

  Suppose G contains a $Q_{3,2}$ with the P_3 given by $a_1c_1c_2a_3$ and the outer C_4 being $a_1a_2a_3a_4$. Then, if a_1c_2 or a_3c_1 or a_2a_4 is an edge, we get a K_4 in G, and if a_1a_3 is an edge, then the 4-cycle $a_1c_1c_2a_3$ along with vertices a_2, a_4 create a $K_{2,2,2}$ in G. Hence, every copy of $Q_{3,2}$ in G has to be induced.

  Suppose $X = V(G) \setminus \{a_1, a_2, a_3, a_4, c_1, c_2\}$, and let $C_i = N_G(c_i) \cap X$ for $i = 1, 2$ and $A_i = N_G(a_i) \cap X$ for $i = 1, \ldots, 4$. Since G is $\widehat{P}_4$-free, we deduce that $A_i \cap A_{i+1} = \varnothing$ and $A_i \cap C_j = \varnothing$ for every i, j (here we denote $A_5 := A_1$), and $C_1 \cap C_2 = \varnothing$. We illustrate this in Figure 5.5, similar to before. Hence, the $Q_{3,2}$'s of G are themselves triangle blocks in G.

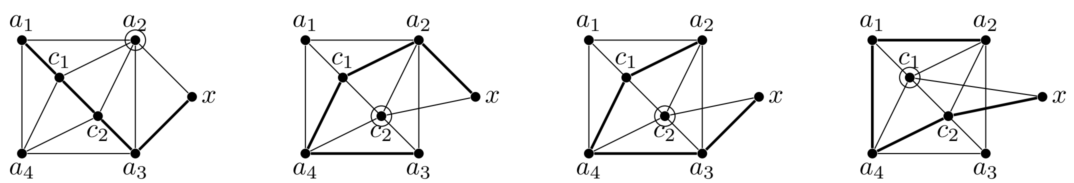

  Figure 5.5: $Q_{3,2}$ is a block by itself.

  Consequently, if G has more than ϵn^2 copies of $Q_{3,2}$, then taking one triangle from each $Q_{3,2}$ we obtain at least ϵn^2 edge-disjoint triangles in G, again contradicting (5.3). We delete an outer edge from each copy of $Q_{3,2}$, losing at most ϵn^2 triangles of G. Hence, we can assume that $t(G) \geq \frac{n^2}{8} + 2\epsilon n^2$, and that G is $\{\widehat{P}_4, K_4, K_{2,2,2}, Q_{3,2}\}$-free.

- **Step 6:** Cleaning $K_{1,2,2}$'s.

  We proceed similarly as before. First, for a $K_{1,2,2}$ with center c and outer cycle $a_1a_2a_3a_4$, we call the edges ca_i "central edges". If G contains such a $K_{1,2,2}$, then one cannot have an edge a_1a_3 or a_2a_4 since these give rise to K_4's through c. Hence, the $K_{1,2,2}$'s in G are induced. Plus, none of the edges a_ic lie in a new triangle since it leads to a $\widehat{P}_4$: they all have codegree 2. We now do a case analysis to see that the $K_{1,2,2}$'s in G are edge-disjoint. Let $A, B \in \binom{V(G)}{5}$ be such that $G[A]$ and $G[B]$ are two $K_{1,2,2}$'s which are not edge-disjoint. Then $2 \leq |A \cap B| \leq 4$. Let the central vertices of $G[A]$ and $G[B]$ be u and v, respectively. Since each central edge of $G[A]$ and $G[B]$ has codegree 2, $u \neq v$.

  Suppose $|A \cap B| = 2$. If $u \in A \cap B$, then the edge through u with its other endpoint in $A \cap B$ must have codegree at least 3. Thus, the central vertices of $G[A]$ and $G[B]$ must lie outside $A \cap B$, leading us to the first configuration in Figure 5.6. But this configuration admits a P_4 in the neighborhood of either vertex of $A \cap B$, a contradiction. We illustrate $G[A \cap B]$ in boldface.

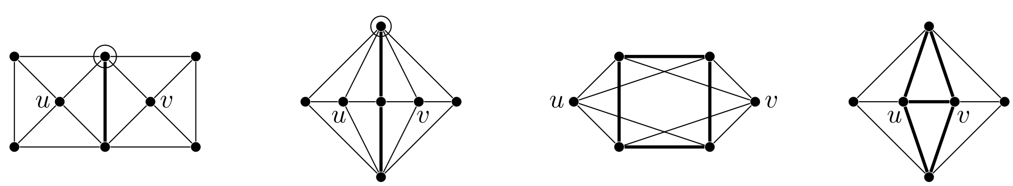

  Figure 5.6: The different ways two induced $K_{1,2,2}$'s can intersect.

  Next, suppose $|A \cap B| = 3$. If $u \in A \cap B$ but $v \notin A \cap B$, then one of the central edges of $G[A]$ contains an external triangle through v. If $u, v \in A \cap B$, then as u is part of the outer C_4 of $G[B]$, uv contains a triangle from $G[B]$ which is not contained in $G[A]$. Thus the only possibility for $|A \cap B| = 3$ is for u and v to be both outside $A \cap B$. This gives rise to the second configuration in Figure 5.6, which contains $\widehat{P}_4$ in the neighborhood of one of the vertices of $A \cap B$.

Finally, if $|A \cap B| = 4$ and $u \in A \cap B$ but $v \notin A \cap B$, then any edge of $G[A \cap B]$ through u has codegree at least 3. If both u and v lie outside $A \cap B$, we obtain a $K_{2,2,2}$, which is the third configuration in Figure 5.6. Hence, u and v must both lie inside $A \cap B$. Since u and v both must be adjacent to all other vertices of $A \cap B$, $G[A \cup B]$ form a $Q_{3,2}$, the fourth configuration in Figure 5.6.

Therefore, if two $K_{1,2,2}$'s are not edge-disjoint, they must intersect each other in one of the ways depicted in Figure 5.6, and we either find a $\widehat{P}_4$, $K_{2,2,2}$ or $Q_{3,2}$ inside G for each of these intersecting patterns. Thus, all $K_{1,2,2}$'s of G are edge-disjoint. Consequently, if G has ϵn^2 copies of $K_{1,2,2}$, they are all edge-disjoint, and give us at least ϵn^2 edge-disjoint triangles, again contradicting (5.3).

For each $K_{1,2,2}$ in G with central vertex x and outer cycle $abcd$, we observe that either ab or bc has codegree 1. Otherwise, suppose $y, z \in V(G)$ are such that yab and zbc form triangles in G. If $y = z$, this creates a $Q_{3,2}$ in G. Otherwise, $yaxcz$ is a P_4 in the neighborhood of b. So, every $K_{1,2,2}$ has an outer edge of codegree 1. By deleting one such edge of codegree 1 from each copy of $K_{1,2,2}$, we remove at most ϵn^2 triangles from G. Therefore, we may assume that G is $\{\widehat{P}_4, K_4, K_{1,2,2}\}$-free, and

$$t(G) \geq \frac{n^2}{8} + \epsilon n^2.$$

Let us now analyze the structure of G. We will prove using induction on $t(H)$, that for any subgraph $H \subseteq G$,

$$t(H) \leq \frac{e(H)}{2}. \tag{5.4}$$

When $t(H) = 1$, $e(H) \geq 3$, proving the base case. Now suppose $t(H) > 1$ for some $H \subseteq G$, and that (5.4) holds for all subgraphs H' with $t(H') < t(H)$. Assume without loss of generality that every edge of H lies in at least one triangle. Call an edge of H *light* if it is contained in a unique triangle from H. Call edges that are not light, *heavy*. We observe that if H contains a triangle with two light edges, then deleting them from H leads to a graph $H' \subsetneq H$ with $t(H') = t(H) - 1$ and $e(H') = e(H) - 2$. Using the induction hypothesis on H', $t(H') \leq e(H')/2$, implying $t(H) \leq e(H)/2$. Hence, we may further assume that H contains no triangle with two light edges.

Lemma 5.4. *Suppose H contains two triangles xuv and yuv intersecting in the edge uv. Then either: (a) xu, yv are light and xv, yu are heavy or: (b) xu, yv are heavy and xv, yu are light.*

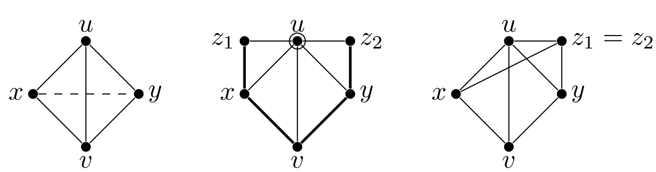

Figure 5.7: xu and yu cannot both be heavy.

Proof of Lemma 5.4. Suppose that both xu and yu were heavy (Figure 5.7). If x and y were adjacent, this would create a K_4 which is forbidden. So, there exist $z_1, z_2 \in V(H)$ such that z_1xu and z_2yu form K_3's in H. If $z_1 \neq z_2$, $N_H(u)$ contains a P_4, which is forbidden. Otherwise $z_1 = z_2$, and this produces a $K_{1,2,2}$ centered at u, a contradiction.

Hence one of xu and yu is light. Similarly, one of xv and yv is light. If xu is light, then xv and yu are heavy, implying that yv is light, and *(a)* holds. Similarly, if xu is heavy, then *(b)* holds.

We shall now use Lemma 5.4 and the fact that every triangle in H has two heavy and one light edge, to analyze the structure of H. First, observe that H cannot have any edge of codegree more than 2. This is because if we have an edge uv which lies in three triangles xuv, yuv, zuv, then by Lemma 5.4, either xu, yv

are light or xv, yu are light. Suppose without loss of generality that xu and yv are light, as in Figure 5.8. Then, by applying Lemma 5.4 on the pairs $\{xuv, zuv\}$ and $\{yuv, zuv\}$ respectively, the edges zv and zu must be light. However, this contradicts the assumption of H containing no triangle with two light edges.

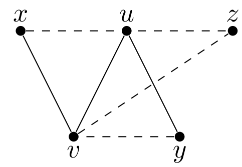

Figure 5.8: Codegree of $uv \in E(H)$ is at most 2.

Now, let $\ell(H)$ denote the number of light edges of H and $h(H)$ the number of heavy edges of H. Since every edge of H can have codegree 1 or 2, and every triangle contains one light and two heavy edges, a double-counting argument gives,

$$\ell(H) + 2h(H) = 3t(H).$$

On the other hand, every light edge of H lies in a unique triangle, and every triangle contains a unique light edge. This implies $t(H) = \ell(H)$. Therefore,

$$4t(H) = 2\ell(H) + 2h(H) = 2e(H),$$

implying $t(H) = \frac{e(H)}{2}$. This finishes the induction step, completing the proof of (5.4).

Taking $H = G$ in (5.4), we obtain $t(G) \leq \frac{e(G)}{2}$. By assumption, $t(G) \geq \frac{n^2}{8} + \epsilon n^2$. So, by Lemma 2.2,

$$\frac{e(G)}{2} \geq t(G) \geq \frac{e(G)}{3n} \cdot (4e(G) - n^2),$$

leading to $e(G) \leq \frac{n^2}{4} + \frac{3n}{8}$. This gives $t(G) \leq \frac{e(G)}{2} \leq \frac{n^2}{8} + \frac{3n}{16}$, which contradicts the assumption of $t(G) \geq \frac{n^2}{8} + \epsilon n^2$ for sufficiently large n. This concludes the proof of Theorem 1.4 for $k = 4$. $\square$

5.2.3 Proof of Theorem 1.4 for k = 5.

Our proof of Theorem 1.4 for $k = 5$ follows exactly the same structure as that for $k = 3$ and $k = 4$, with more technical details. We shall prove, using induction on n, that if G is $\widehat{P}_5$-free, then

$$t(G) \leq \frac{n^2}{4} + 5n. \tag{5.5}$$

The base case $n = 3$ is clearly true as $t(G) \leq 1$. Assume that (5.5) holds for all graphs G on less than n vertices, and let us prove that it also holds for G. The first step is to remove all copies of K_6 and K_6^- from G via the induction hypothesis.

Suppose G has a copy of K_6 with vertices $a_1, \dots, a_6$. Then this is a block by itself, since if there is a vertex $x \neq a_i$ such that xa_1a_2 is a triangle, then $N_G(a_1)$ contains the 5-path $a_6a_5a_4a_3a_2x$. For $1 \leq i \leq 6$, let $X_i = N_G(a_i) \setminus \{a_1, \dots, a_6\}$. Then $X_i \cap X_j = \varnothing$ for every $i \neq j$. Since $\sum_{i=1}^{6} |X_i| \leq n - 6$, there is a vertex a_i for which $|X_i| \leq \frac{n-6}{6}$. By Lemma 2.4,

$$e(X_i) \leq \frac{5-1}{2} \cdot |X_i| \leq \frac{n-6}{3}.$$

Hence, by (5.5) on $G' = G - \{a_i\}$, we get $t(G') \leq \frac{(n-1)^2}{4} + 5(n-1)$. Therefore,

$$t(G) \leq t(G') + \frac{n-6}{3} + 5 \leq \frac{(n-1)^2}{4} + 5(n-1) + \frac{n-6}{3} + 5 < \frac{n^2}{4} + 5n,$$

completing the induction step for G. Hence, we may assume that G is K_6-free.

Now, if G has a copy of K_6^- on vertices $a_1, \ldots, a_6$, it has to be induced. We verify in Figure 5.9 that it is a block by finding a $\widehat{P}_5$ whenever any edge lies in an external triangle. Let $X_i = N_G(a_i) \setminus \{a_1, \ldots, a_6\}$. Following exactly the same argument as before, there exists a vertex a_i for which $|X_i| \leq \frac{n-6}{6}$. Therefore, applying Lemma 2.4 and letting $G' = G - \{a_i\}$,

$$t(G) \leq t(G') + \frac{n-6}{3} + 5 < \frac{n^2}{4} + 5n,$$

completing the induction step for G.

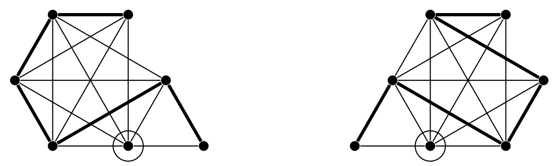

Figure 5.9: K_6^- is a block by itself.

Therefore, without loss of generality we can assume that G is $\{K_6^-, \widehat{P}_5\}$-free. Consider G to be a fixed n-vertex graph. We shall now prove using induction on $t(H)$, that for any subgraph $H \subseteq G$,

$$t(H) \leq e(H). \tag{5.6}$$

When $t(H) = 1$, $e(H) \geq 3$ proves the base case. Now suppose $t(H) > 1$ for some $H \subseteq G$, and that (5.6) holds for all subgraphs H' of G with $t(H') < t(H)$. If H has an edge e which lies in at most one triangle, using the induction hypothesis on $H' = H - \{e\}$ immediately proves (5.6) for H. Hence, we may assume that all edges of H have codegree at least 2. Call an edge of H *light* if it has codegree exactly 2, otherwise call it *heavy*.

Lemma 5.5. *We may assume that H does not contain W_5 and $K_{1,2,2}$ as subgraphs.*

This lemma is proved by sequentially removing copies of the graphs illustrated in Figure 5.10 from H, and the proof can be found in the Appendix. We shall now assume that Lemma 5.5 is true.

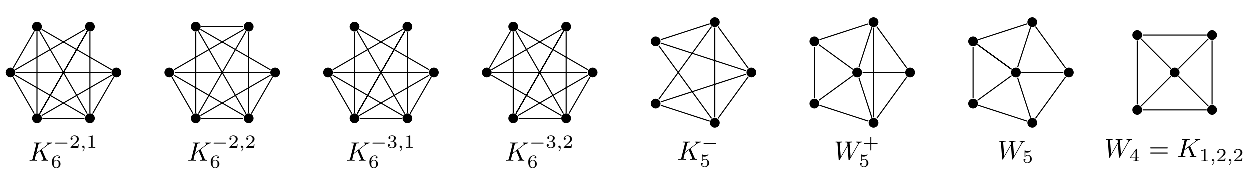

Figure 5.10: Graphs to be cleaned from H.

Suppose H contains a triangle abc such that abx and acy are triangles, with $x \neq y$, i.e H contains a $\widehat{P}_3$. Refer to Figure 5.11. Observe that both ax and ay must have codegree at least 2. If xy is an edge in H, we get a W_4. If axz and ayw are triangles for vertices z and w which are not b or c, then there are two possibilities. Either $z \neq w$ in which case we get a $\widehat{P}_5$, or $z = w$, producing a W_5 in H. Therefore $\{z, w\} \cap \{b, c\} \neq \varnothing$. If $z = c$ and $w = b$, this gives us a $K_{1,2,2}$ centered at a. Hence we may assume $z = c$ and $w \neq b$. By assumption, aw must have codegree at least 2. Note that wb or wx cannot be edges, as they create $K_{1,2,2}$ or W_5 in H centered around a, respectively. Further, we cannot have a new vertex t for which awt is a triangle, since this creates a $\widehat{P}_5$ centered at a. Thus, the only possibility is that $wc \in E(H)$.

As H is $\{K_{1,2,2}, W_5\}$-free, $H[a, b, c, x, y, w]$ is induced. Further, if the edges ax, ab, ay, aw or cb, cx, cw, cy lie in an external triangle, we can find $\widehat{P}_5$'s centered around a or c, respectively. Hence these 8 edges all do not

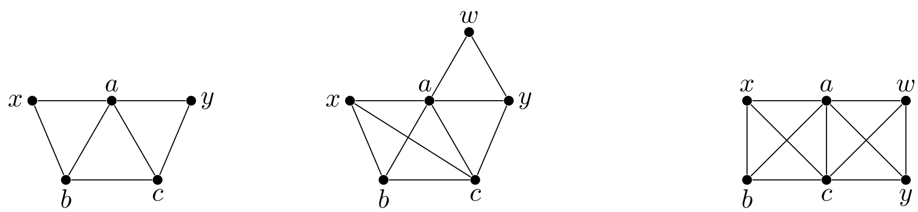

Figure 5.11: abc, abx, acy are triangles with $x \neq y$.

lie in external triangles, and have codegree exactly 2. Deleting them from H, we obtain a graph H' with $t(H') = t(H) - 8$ and $e(H') = e(H) - 8$, completing the proof of (5.6) for H.

Hence, we may assume that H does not contain any $\widehat{P}_3$. Now if ab and ac were heavy in any triangle abc, we would then find $x \neq y$ for which abx and acy are triangles in H. This leads us to the following crucial observation:

$$\text{Every triangle of } H \text{ has at most one heavy edge.} \tag{5.7}$$

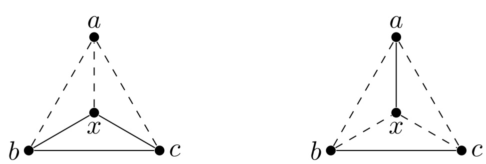

Figure 5.12: Edge xa can be light or heavy.

Let us fix a triangle abc in H. Let ab and ac be light. As they must have codegree 2, there is a vertex x for which $xa, xb, xc \in E(H)$, as in Figure 5.12. If the edge xa is light, we can then let $H' = H - \{ab, ax, ac\}$. Note that $t(H') = t(H) - 3$ and $e(H') = e(H) - 3$, finishing the proof of (5.6) for H. Finally, if xa is heavy, then by (5.7), the edges xb and xc must be light. Let $H' = H - \{ab, ac, xb, xc\}$, then $t(H') = t(H) - 4$ and $e(H') = e(H) - 4$, completing the induction step of (5.6).

Taking $H = G$ in (5.6), we obtain $t(G) \leq e(G)$. Using Lemma 2.2,

$$e(G) \geq t(G) \geq \frac{e(G)}{3n} \cdot (4e(G) - n^2),$$

implying $t(G) \leq e(G) \leq \frac{n^2}{4} + \frac{3n}{4} < \frac{n^2}{4} + 5n$. This concludes the proof of (5.6) for G.

6 Concluding Remarks

The following generalization of Theorems 1.1 and 1.2 was obtained by an anonymous referee during the review process.

Theorem 6.1. *Suppose H is a graph with $ex(n, H) = O(n^\alpha)$ for some $1 < \alpha < 2$. Then, $ex(n, K_3, \widehat{H}) = o(n^{1+\alpha})$.*

Proof of Theorem 6.1. Recall that H is a graph such that $\mathrm{ex}(n, H) \leq Cn^\alpha$, where $C > 0, 1 < \alpha < 2$, and we want to show that $\mathrm{ex}(n, K_3, \widehat{H}) = o(n^{1+\alpha})$. First, let us fix a graph G on n vertices that is $\widehat{H}$-free, and any $\epsilon > 0$. The number of triangles in G is at most $n \cdot \mathrm{ex}(n, H) \leq Cn^{1+\alpha} = o(n^3)$ many triangles. Thus, by Lemma 2.1, every maximal collection of edge-disjoint triangles in G has size at most ϵn^2. Hence, there is a set F of edges of G such that $|F| = 3\epsilon n^2$ and every triangle of G has an edge in F. For every vertex

$v \in V(G)$, define $N_F(v) := \{x \in V(G) : vx \in F\} \subseteq N_G(v)$, and $N_R(v) := N_G(v) \setminus N_F(v)$. Since every triangle of G is incident to $N_F(v)$ for some $v \in V(G)$,

$$t(G) \leq \sum_{v\in V(G)} e(N_F(v)) + \sum_{v\in V(G)} e(N_F(v), N_R(v)). \tag{6.1}$$

Here $e(N_F(v), N_R(v))$ denotes the number of edges between vertex subsets $N_F(v)$ and $N_R(v)$ in G. Now we bound each term in (6.1). By assumption, as $N_G(v)$ is H-free, we have

$$e(N_F(v)) \leq \mathrm{ex}(|N_F(v)|, H) \leq C|N_F(v)|^\alpha \leq Cn|N_F(v)|^{\alpha-1}.$$

Further, we can partition $N_R(v)$ into $r = \left\lceil \frac{|N_R(v)|}{|N_F(v)|} \right\rceil$ sets $X_1, \dots, X_r$ with $|X_i| \leq |N_F(v)|$ for every $i \in [r]$. Then,

$$\begin{aligned} e(N_F(v), N_R(v)) = \sum_{i=1}^{r} e(N_F(v), X_i) &\leq \sum_{i=1}^{r} \mathrm{ex}(2|N_F(v)|, H) \\ &\leq C \cdot r \cdot (2|N_F(v)|)^\alpha \\ &\leq C \cdot \frac{2|N_R(v)|}{|N_F(v)|} \cdot 2^\alpha |N_F(v)|^\alpha \\ &\leq 2^{1+\alpha} Cn|N_F(v)|^{\alpha-1}. \end{aligned}$$

Therefore, (6.1) gives us

$$t(G) \leq \sum_{v\in V(G)} \left((2^{1+\alpha}+1)C \cdot n|N_F(v)|^{\alpha-1}\right) = C'n \sum_{v\in V(G)} |N_F(v)|^{\alpha-1},$$

where $C' = (2^{1+\alpha}+1)C$. Observe that $\sum_{v\in V(G)} N_F(v) = 2|F| = 6\epsilon n^2$. Now, as $1 < \alpha < 2$, the function $f(x) = x^{\alpha-1}$ is concave in x, and thus by Jensen's inequality,

$$t(G) \leq C'n^2 \cdot \left(\frac{1}{n} \sum_{v\in V(G)} |N_F(v)|\right)^{\alpha-1} = C'n^2 \cdot (6\epsilon n)^{\alpha-1} = C''\epsilon^{\alpha-1} n^{1+\alpha},$$

where $C'' = C' \cdot 6^{\alpha-1}$. As $\epsilon > 0$ was arbitrary, this implies $t(G) = o(n^{1+\alpha})$.

□

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7 Appendix

Our goal in this section is to complete the proof of Lemma 5.5. Recall that H is a subgraph of a $\{K_6^-, \widehat{P}_5\}$-free graph G such that every edge of H has codegree at least 2.

Proof of Lemma 5.5. We wish to show that H does not contain copies of W_5 or $K_{1,2,2}$. We do this via sequentially cleaning the following graphs from H:

- $K_6^{-2,1}$, the graph obtained from K_6 by deleting two intersecting edges,
- $K_6^{-2,2}$, the graph obtained from K_6 by deleting two non-intersecting edges,
- $K_6^{-3,1}$, the graph obtained from K_6 by deleting a P_3,
- $K_6^{-3,2}$, the graph obtained from K_6 by deleting a $P_2 \sqcup K_2$,
- K_5,
- K_5^-, the graph obtained from K_5 by deleting one edge,
- W_5^+, the graph obtained from the 5-wheel graph $W_5 = \widehat{C}_5$ by adding an edge,
- W_5, and
- $K_{1,2,2}$, the 4-wheel graph.

More specifically, whenever H contains a copy of one of these graphs, we would be able to use the induction hypothesis of (5.6) on some subgraph $H' \subsetneq H$ and complete the induction step for H.

Before proceeding onto the cleaning steps, we make an important observation:

$$\text{If } abcde \text{ is a } P_4 \text{ in } N_H(x), \text{ then } N_H(\{a,x\}) \subseteq \{b,c,d,e\}. \tag{7.1}$$

This is because since xa has codegree at least 2, we must have a vertex $y \in V(H)$ with xay being a triangle, $y \neq b$. If $y \notin \{c,d,e\}$, then we get a $\widehat{P}_5$ centered around x. Thus $y = c$ or $y = d$ or $y = e$, implying (7.1).

1. Cleaning $K_6^{-2,1}$: Suppose H has a copy of $K_6^{-2,1}$ with vertices $\{a,b,c,d,e,f\}$ such that the edges ab and bc are missing. This is an induced subgraph as H has no K_6^-. Further, all edges of this subgraph other than ac cannot belong to external triangles, as verified in Figure 7.1.

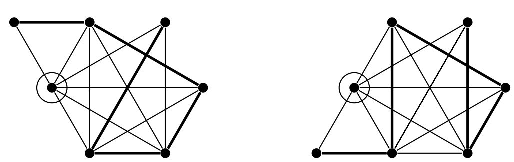

Figure 7.1: All edges but ac cannot lie in external triangles.

Now suppose ac lies on an external triangle, acx. By (7.1) on the 4-path $fedcx$ in $N_H(a)$, $N_H(\{a,x\}) \subseteq \{c,d,e\}$. Moreover, ax has codegree at least 2. Thus, xd, xe, or xf is an edge of H. In either case we obtain $\widehat{P}_5$'s in H, as shown in Figure 7.2.

Hence $K_6^{-2,1}$ is a block by itself. Let H' be the subgraph of H obtained by deleting all edges from this copy of $K_6^{-2,1}$. Then note that $t(H') = t(H) - 13$ and $e(H') = e(H) - 13$, completing the induction step for H.

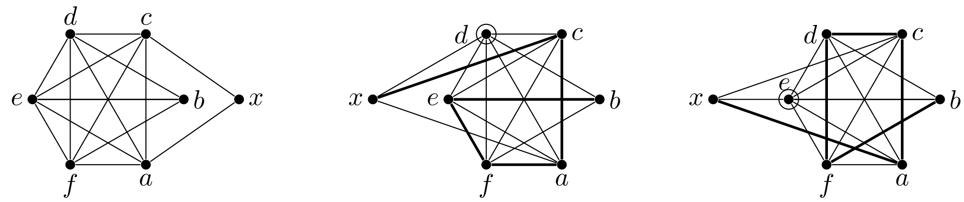

Figure 7.2: acx is a triangle.

2. Cleaning $K_6^{-2,2}$: Let H have a copy of $K_6^{-2,2}$ with vertices $\{a,b,c,d,e,f\}$ such that the edges ab and cd are missing. Clearly this is an induced subgraph of H. It can be checked that the edges ea, ec, eb, ed cannot lie on external triangles as otherwise we would get $\widehat{P}_5$'s centered at e. Similarly, the edges fc, fa, fd, fb cannot lie on external triangles.

Now, suppose the edge ad lies in an external triangle adx. Refer to Figure 7.3. As $cfedx$ is a P_4 in the neighborhood of a, (7.1) implies that $N_H(\{a,x\}) \subseteq \{d,e,f,c\}$ and $N_H(\{a,c\}) \subseteq \{f,e,d,x\}$. If $xe \in E(H)$, this leads to a $\widehat{P}_5$ centered at e, given by the 5-path $caxdbf$. If $xf \in E(H)$, we get a $\widehat{P}_5$ centered at f, given by the 5-path $bdxac$. Thus, $xc \in E(H)$, and the edge xa is light. Repeating the same argument for the 4-path $bfeax$ around d, we get $xb \in E(H)$, xd is light and ac has codegree 3. Using (7.1) on the 4-path $xaefb$ in $N_H(c)$ and $N_H(d)$ respectively, we get $N_H(\{c,x\}) \subseteq \{a,e,f,b\}$ and $N_H(\{b,d\}) \subseteq \{f,e,a,x\}$. Since we already know that $xe, xf, ab \notin E(H)$, this means that the edge xc is light and bd has codegree 3. Similarly, bx is light. Now, let

$$H' = H - \{ea, ec, eb, ed, fc, fa, fd, fb, xa, xc, xb, xd, ac, bd\}.$$

Clearly $e(H') = e(H) - 14$ and $t(H') = t(H) - 14$, and we are done by induction.

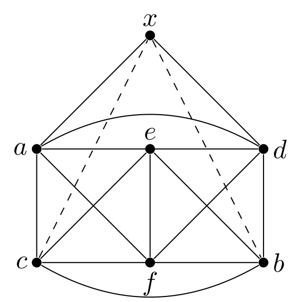

Figure 7.3: Edge ad lies in triangle adx, $x \notin \{a,b,c,d,e,f\}$.

Therefore, ad cannot lie in any external triangle adx, and is light. Similarly, bc is light. Let $H' = H - \{ea, ec, eb, ed, fc, fa, fd, fb, ad, bc\}$. Then $t(H') = t(H) - 10$ and $e(H') = e(H) - 10$, finishing the induction step for H.

3. Cleaning $K_6^{-3,1}$: Since G has no $K_6^{-2,1}$ or $K_6^{-2,2}$ which are the only two ways one can delete two edges from K_6, any copy of $K_6^{-3,1}$ is induced. Suppose such a copy of $K_6^{-3,1}$ exists in G, and is given by the complete graph on $\{a,b,c,d,e,f\}$ minus the edges $\{ab, bc, cd\}$. By an argument exactly the same as before, $ea, ec, eb, ed, fc, fa, fd, fb$ are light. Further, if ad lies in an external triangle adx (as in Figure 7.4), then by repeating the argument for cleaning $K_6^{-2,2}$, we note that $xc, xb \in E(H)$, xa, xd are light, and ac, bd have codegree exactly three. Also, by using (7.1) on the 4-path $dxcfe$, we have $N_H(\{a,d\}) \subseteq \{x,c,f,e\}$. As $cd \notin E(H)$, this means $N_H(\{a,d\}) = \{x,f,e\}$, and therefore ad has codegree three. Finally, (7.1) on the path $cfedx$ in $N_H(a)$ gives us $N_H(\{a,c\}) \subseteq \{f,e,d,x\}$, whereas $cd \notin E(H)$, implying that ac has codegree three as well. Similarly, bd has codegree three.

Let $H' = H - \{ea, ec, eb, ed, fc, fa, fd, fb, ac, bd, ad, xa, xd\}$. Then, $t(H') = t(H) - 13$ and $e(H') = e(H) - 13$, and we can proceed by the induction hypothesis on H'.

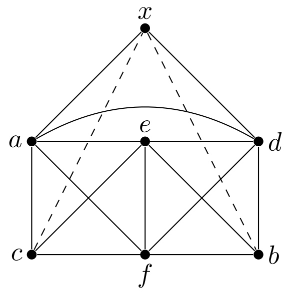

Figure 7.4: adx is an external triangle, ab, bc, ca are non-edges in H.

On the other hand, if the edge ad is light, then we can simply let $H' = H - \{ea, ec, eb, ed, fc, fa, fd, fb, ad\}$, whence $t(H') = t(H) - 9$ and $e(H') = e(H) - 9$, and the induction step would be complete. Hence we can assume that H is $K_6^{-3,1}$-free.

4. Cleaning $K_6^{-3,2}$: Suppose H contains a $K_6^{-3,2}$ on vertices $\{a, b, c, d, e, f\}$ such that edges ab, cd, de are missing. Since H is $K_6^{-2,1}$ and $K_6^{-2,2}$-free, this subgraph is induced. As the edges bd and ad must have codegree at least two, there exist $x, y \in V(H) \setminus \{a, b, c, d, e, f\}$ such that bdx and ady are triangles in H. We consider two different cases.

- **Case 1.** $x = y$ (Figure 7.5 (left)): Since $N_H(b)$ contains the 4-path $cefdx$, (7.1) gives $N_H(\{b, x\}) \subseteq \{d, f, e, c\}$. If $xf \in E(H)$, then $N_H(f)$ contains the 5-path $xadbce$. If both xc and xe were edges in H, then $H[\{a, b, c, e, f, x\}]$ would be a $K_6^{-2,2}$ with edges xf, ab missing. Therefore, only one of xc and xe can be an edge. By symmetry, assume $xc \in E(H)$ and $xe \notin E(H)$.

  As this fixes edges and non-edges between any pair of vertices from $\{a, b, c, d, e, f, x\}$, $H[\{a, b, c, d, e, f, x\}]$ is induced. Consider the 5-wheel $(f, adbcea)$, where the first tuple denotes the central vertex and the second tuple is the outer C_5. Since none of the edges fa, fb, fc, fd, fe can lie in triangles with a vertex $y \notin \{a, b, c, d, e, f, x\}$ (it would give a $\widehat{P}_5$ around f), they all have exhausted their codegrees. Similarly, $(c, befaxb)$, $(b, cxdfec)$, and $(a, dxcefd)$ are W_5's in H. Let

  $$H' = H - \{cb, cx, ca, cf, ce, fe, fb, fd, fa, be, bx, bd, ad, ax, ae\}.$$

  Then, $e(H') = e(H) - 15$ and $t(H') = t(H) - 13$ (4 triangles through x, 7 through f but not x, and 2 not through x or f). We can then proceed with the induction hypothesis on H'.

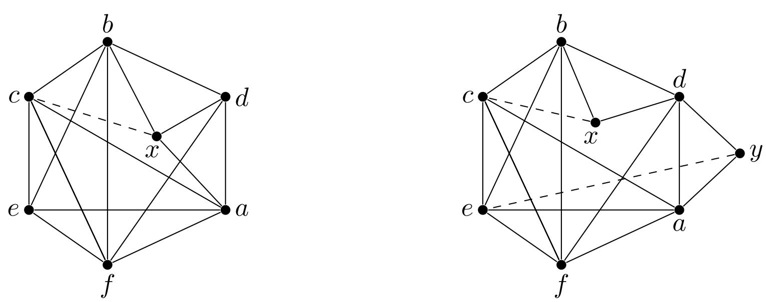

Figure 7.5: bdx and ady are triangles, (left: $x = y$, right: $x \neq y$).

- **Case 2.** $x \neq y$ (Figure 7.5 (right)): Without loss of generality assume $by, ax \notin E(H)$, as these would lead us to Case 1. As $N_H(b)$ contains the 4-path $xdfec$, by (7.1), $N_H(\{b, x\}) \subseteq \{d, f, e, c\}$. Note that if $xf \in E(H)$, then $N_H(f)$ contains the path $ecbxda$ of length 5. Hence, $N_H(\{b, x\}) \subseteq \{d, e, c\}$. Further, both xc and xe cannot be edges in H, as then $H[\{a, b, c, e, f, x\}] \supseteq K_6^{-3,1}$ with the edges ba, ax, xf missing. As codegree of bx is at least 2, exactly one of xc and xe is an edge in H. By symmetry, assume $xc \in E(H)$ and $xe \notin E(H)$.

  Now by (7.1) on the 4-path $ydfec$ in $N_H(a)$, we get $N_H(\{a, y\}) \subseteq \{d, f, e, c\}$. If $yf \in E(H)$, then $N_H(f)$ contains the path $aydbce$ of length 5, and if $yc \in E(H)$, then $N_H(c)$ contains the path $xbefay$ of length 5. Therefore, $N_H(\{a, y\}) = \{d, e\}$, and $ye \in E(H)$.

  Finally, let us consider the 4-path $yafbc$ in $N_H(e)$. Using (7.1),

  $$N_H(\{y, e\}) \subseteq \{a, f, b, c\}.$$

  However, $yf, yc \notin E(H)$ from our argument in the last paragraph, and $by \notin E(H)$ as we are in Case 2. This is a contradiction, as the edge ye must have codegree at least 2.

5. Cleaning K_5: If H contains a copy of K_5 on vertex set $\{a, b, c, d, e\}$, then we claim that it is a block by itself. Suppose $x \in V(H) \setminus \{a, b, c, d, e\}$ is such that abx is a triangle in H. Since $xbcde$ is a P_4 in $N_H(a)$, (7.1) implies that $N_H(\{a, x\}) \subseteq \{b, c, d, e\}$. Further, ax must have codegree at least 2. Thus, xc, xd, or xe is an edge. In either case, $H[\{a, b, c, d, e, x\}] \supseteq K_6^{-2,1}$, a contradiction.

6. Cleaning K_5^-: Let H have a copy of K_5^- on vertices a, b, c, d, e such that $ab \notin E(H)$. If the edge bc lies in an external triangle bcx as shown in Figure 7.6, then note that xb has codegree at least two, and (7.1) on the 4-path $xbdea$ in $N_H(c)$ tells us that $N_H(\{c, x\}) \subseteq \{b, d, e, a\}$. If $xe \in E(H)$ then $G[a, b, c, d, e, x]$ contains the graph $K_6^{-3,1}$ with edges dx, xa, ab missing. If $xd \in E(H)$, then we have the $K_6^{-3,1}$ with edges ex, xa, ab missing. Finally, if $xa \in E(H)$, then $G[a, b, c, d, e, x]$ contains $K_6^{-3,2}$ with edges ex, xd, ab missing. Thus, the edge bc cannot lie on an external triangle.

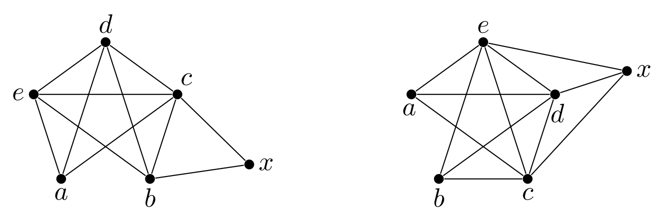

Figure 7.6: K_5^- is a block by itself.

Thus by symmetry, ae, ad, ac, be, bd, bc cannot lie on external triangles. Now suppose that the edge cd lies on an external triangle cdx. By (7.1) on any $\widehat{P}_4$ centered at c, $N_H(\{c, x\}) \subseteq \{d, e, a, b\}$. If either xa or xb is an edge, we obtain a $K_6^{-3,1}$ with missing edges ab, bx, xe or ba, ax, xe, respectively. So assume $xa, xb \notin E(H)$. Thus $ex \in E(H)$, and cx has codegree 2. Similarly, dx has codegree 2. Now using (7.1) on the 4-path $xdbca$ in $N_H(e)$, we have $N_H(\{e, x\}) \subseteq \{a, b, c, d\}$. Since $xa, xb \notin E(H)$, ex must have codegree 2. Thus, the edges xc, xd, xe all have codegree 2. Let $H' = H - \{xc, xd, xe\}$, then $t(H') = t(H) - 3$ and $e(H') = e(H) - 3$, and we can proceed by induction.

Hence, we may assume that cd also does not lie on external triangles. Let $H' = H - \{ac, ad, ae, bc, bd, be, cd\}$. Then, $t(H') = t(H) - 7$ and $e(H') = e(H) - 7$, completing the induction hypothesis for H again. So, without loss of generality we can assume that G is $\{K_5^-, \widehat{P}_5\}$-free.

7. Cleaning W_5^+: Let H contain a W_5^+, given by central vertex x, outer cycle $abcde$ with an edge $ac \in E(H)$. If $ad \in E(H)$, then a, b, c, d, x form a K_5^- in H. Therefore by symmetry, all copies of W_5^+ in H are induced.

Now let us fix such a W_5^+ in H with vertices labeled as above. As cd and ae must have codegree at least 2, there exist $y, z \in V(H) \setminus \{a, b, c, d, e, x\}$ such that aey and cdz are triangles in H. Then, we have two possible cases:

- **Case 1:** $y = z$. Refer to Figure 7.7. Note that if $yx \in E(H)$, then $N_H(x)$ would contain the P_5 given by $yabcde$. Hence $yx \notin E(H)$. Using (7.1) on the 4-path $yexcb$, $N_H(\{a, y\}) \subseteq \{e, x, c, b\}$. Suppose $yb \in E(H)$. Note that $(y, abcdea)$, $(c, aydxba)$, $(a, cybxec)$ form W_5^+'s in H. As (7.1) together with the fact that every W_5^+ of H is induced imply that each central edge of any copy of W_5^+ in H does not lie on external triangles, the edges xa, xb, xc, xd, xe; ya, yb, yc, yd, ye; cb, ca, cd; ab, ae cannot lie in external triangles. Thus, we can delete these 15 edges from H, and only lose 13 triangles (6 through x, 6 through y, and the triangle abc). Our proof would then be complete by induction.

  Hence, assume $yb \notin E(H)$. From $N_H(\{a, y\}) \subseteq \{e, x, c, b\}$ this implies that codegree of ay is exactly 2. Similarly, cy has codegree 2. Further, (7.1) on the path of length four $bxcye$ in $N_H(a)$ implies that $N_H(\{a, b\}) \subseteq \{x, c, y, e\}$. Since $by, be \notin E(H)$, this implies that $N_H(\{a, b\}) = \{x, c\}$, and ab has codegree 2. Similarly, bc has codegree 2. Let $H' = H - \{xa, xb, xc, xd, xe, ab, bc\}$. Then, $t(H') = t(H) - 7$ and $e(H') = e(H) - 7$, again concluding the induction step.

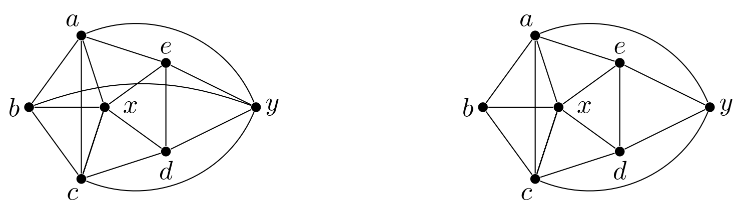

Figure 7.7: Case 1: $y = z$, aey and cdy are triangles in H.

- **Case 2:** $y \neq z$. Since $N_H(a)$ contains the path $yexcb$ of length 4, we must have $N_H(\{a, y\}) \subseteq \{e, x, b, c\}$. If $yx \in E(H)$, we obtain the 5-path $yabcde$ in $N_H(x)$, and if $yc \in E(H)$ we obtain the 5-path $yabxdz$ in $N_H(c)$. On the other hand, ay must have codegree at least 2. Thus $N_H(\{a, y\}) = \{b, e\}$ and ay is light. By a symmetric argument, $zb \in E(H)$ and cz is also light. Refer to Figure 7.8. Since $N_H(b)$ contains the P_4 given by $zcxay$, we have $N_H(\{b, y\}) \subseteq \{a, x, c, z\}$. However, we have already observed that $yx, yc \notin E(H)$. Hence $zy \in E(H)$, and by is light. Similarly, bz is light.

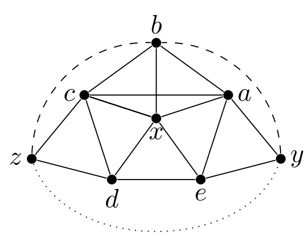

Figure 7.8: Case 2: $y \neq z$, aey and cdz are triangles in H.

  Now, observe that we have produced two W_5^+'s given by $(c, zdxabz)$ and $(a, yexcby)$, both with the extra edge ac. By (7.1) in $N_H(c)$ and $N_H(b)$, all of the central edges cannot lie in external triangles. Let
$$H' = H - \{cz, cd, cx, cb, ay, ae, ax, ab, ac, by, bz\}.$$

It is clear that $e(H') = e(H) - 11$, and that deleting these edges, we delete 6 triangles through c and 4 triangles through a that do not contain c, and the triangle byz through b which does not contain a or c. Hence, $t(H') = t(H) - 11$. This completes our induction step.

We may therefore assume that H is W_5^+-free.

8. Cleaning W_5: Suppose H has a copy of W_5 given by $(x, abcdea)$. As H is W_5^+-free, $H[\{a, b, c, d, e, x\}] \cong W_5$. By (7.1) applied to $N_H(x)$, every central edge is light. Thus, we may let $H' = H - \{xa, xb, xc, xd, xe\}$, whence $t(H') = t(H) - 5$ and $e(H') = e(H) - 5$, allowing us to complete the induction step for H.

9. Cleaning $K_{1,2,2}$: Finally, let H contain a $K_{1,2,2}$ with central vertex x and outer cycle $abcd$. Since H is K_5^--free, $H[\{a, b, c, d, x\}] \cong K_{1,2,2}$. We claim that none of the edges xa, xb, xc, xd lie on an external triangle.

For the sake of contradiction, assume $y \in V(H) \setminus \{a, b, c, d, x\}$ is such that xay is a triangle in H (Figure 7.9). By (7.1) in $N_H(a)$, we have $N_H(\{a, y\}) \subseteq \{x, d, c, b\}$. If $yd \in E(H)$, we obtain the 5-wheel $(x, ydcbay)$, and if $yb \in E(H)$, we obtain the 5-wheel $(x, yadcby)$. Since $|N_H(\{a, y\})| \geq 2$, we must have $N_H(\{a, y\}) = \{x, c\}$. But then, $H[\{c, x, b, y, a\}] \cong K_5^-$, a contradiction.

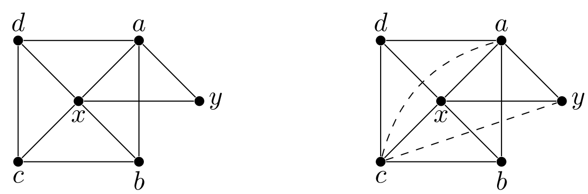

Figure 7.9: xa cannot lie in an external triangle xay.

Hence, the edges xa, xb, xc, xd all have codegree 2. Let $H' = H - \{xa, xb, xc, xd\}$, then $t(H') = t(H) - 4$ and $e(H') = e(H) - 4$, finishing the induction step in this case as well.

Hence, after these cleaning procedures, we may assume that H is a $\{W_5, K_{1,2,2}\}$-free subgraph of G such that every edge of H has codegree at least 2. This concludes the proof of Lemma 5.5. $\square$