Eigenvalues and Linear Quasirandom Hypergraphs

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September 5, 2013

Abstract

Let $p(k)$ denote the partition function of k. For each $k \geq 2$, we describe a list of $p(k) - 1$ quasirandom properties that a k-uniform hypergraph can have. Our work connects previous notions on linear hypergraph quasirandomness of Kohayakawa-Rödl-Skokan and Conlon-Hàn-Person-Schacht and the spectral approach of Friedman-Wigderson. For each of the quasirandom properties that are described, we define a largest and second largest eigenvalue. We show that a hypergraph satisfies these quasirandom properties if and only if it has a large spectral gap. This answers a question of Conlon-Hàn-Person-Schacht. Our work can be viewed as a partial extension to hypergraphs of the seminal spectral results of Chung-Graham-Wilson for graphs.

1 Introduction

The study of quasirandom or pseudorandom graphs was initiated by Thomason [43, 44] and then refined by Chung, Graham, and Wilson [15], resulting in a list of equivalent (deterministic) properties of graph sequences which are inspired by $G(n, p)$. Beginning with these foundational papers on the subject [15, 43, 44], the last two decades have seen an explosive growth in the study of quasirandom structures in mathematics and computer science. For details on quasirandomness, we refer the reader to a survey of Krivelevich and Sudakov [30] for graphs and recent papers of Gowers [23, 24, 25] for general quasirandom structures including hypergraphs.

1.1 Previous Results

The core of what Chung, Graham, and Wilson [15] proved is that several properties of graph sequences are equivalent. Two of them are `Disc` and `Count[All]`. The first states that all sufficiently large vertex sets have the same edge density as the original graph and the second states that for all fixed graphs F the number of copies of F is what one would expect in a random graph with the same density.

A k-uniform hypergraph is a pair of finite sets $(V(H), E(H))$ where $E(H) \subseteq \binom{V(H)}{k}$ is a collection of k-subsets of $V(H)$. For $U \subseteq V(H)$, the induced subgraph on U, denoted $H[U]$, is the hypergraph with vertex set U and edge set $\{e \in E(H) : e \subseteq U\}$. If F and G are hypergraphs, a labeled copy of F in H is an edge-preserving injection $V(F) \to V(H)$, i.e. an injection $\alpha : V(F) \to V(H)$ such that if E is an edge of F, then $\{\alpha(x) : x \in E\}$ is an edge of H. A graph is a 2-uniform hypergraph.

Almost immediately after proving their theorem, Chung and Graham [8, 9, 12, 13, 14] investigated generalizing the theorem to k-uniform hypergraphs. One initial difficulty in generalizing quasirandomness to $k > 2$ is an observation by Rödl that a construction of Erdős and Hajnal [18] shows that the hypergraph generalizations of `Disc` and `Count[All]` are not equivalent. Motivated by this, Chung and Graham [9, 12, 13, 14] investigated how to strengthen the property `Disc` to make it equivalent to `Count[All]`. They found several properties equivalent to `Count[All]`; the main property they use is related to the number of even/odd subgraphs of a given hypergraph which they called `Deviation`. Simultaneously, Frankl and Rödl [20] also obtained a property stronger than `Disc` which is equivalent to `Count[All]`. Subsequently, other properties equivalent to `Count[All]` have been studied by several researchers [6, 23, 27, 29].

It remained open whether the simpler property `Disc` for k-uniform hypergraphs is equivalent to counting some class of hypergraphs or counting a single substructure. This is related to the Weak Hypergraph Regularity Lemma [10, 20, 39]. Recently, Kohayakawa, Nagle, Rödl, and Schacht [28] answered this question by showing that `Disc` is equivalent to counting the family of linear hypergraphs, where a hypergraph H is linear if every pair of distinct edges share at most one vertex. Building on this, Conlon, Hàn, Person, and Schacht [16] showed that `Disc` is equivalent to counting a type of linear four cycle. These two results can be combined into the following theorem.

Theorem 1. *(Kohayakawa-Nagle-Rödl-Schacht [28] and Conlon-Hàn-Person-Schacht [16]) Let $0 < p < 1$ be a fixed constant and let $\mathcal{H} = \{H_n\}_{n \to \infty}$ be a sequence of k-uniform hypergraphs such that $|V(H_n)| = n$ and $|E(H_n)| \geq p\binom{n}{k} + o(n^k)$. The following properties are equivalent:*

- *`Disc`: For every $U \subseteq V(H_n)$, $|E(H_n[U])| = p\binom{|U|}{k} + o(n^k)$.*

- *`Count[linear]`: For every fixed linear k-uniform hypergraph F with e edges and f vertices, the number of labeled copies of F in H_n is $p^e n^f + o(n^f)$.*

- *`Cycle`$_4$: The number of labeled copies of C_4 in H_n is at most $p^{|E(C_4)|} n^{|V(C_4)|} + o(n^{|V(C_4)|})$, where C_4 is a linear hypergraph defined precisely in Section 2.*

Note that Conlon et al. [16] put the condition "$|E(H_n)| \geq p\binom{n}{k} + o(n^k)$" into the statement of the properties that don't trivially imply it like `Disc` and this is equivalent to the way we have stated Theorem 1. Conlon et al. [16] have several more properties including induced subgraph counts and common neighborhood sizes, but we consider the properties stated in Theorem 1 as the core properties.

1.2   Our Results

Another graph property equivalent to `Disc` is `Eig`, which states that if μ_1 and μ_2 are the first and second largest (in absolute value) eigenvalues of the adjacency matrix of the graph respectively, then $\mu_2 = o(\mu_1)$. Neither Chung and Graham [9, 12, 13, 14] nor Kohayakawa, Rödl, and Skokan [29] provided a generalization of `Eig` to hypergraphs. Later, Conlon, Hàn, Person, and Schacht [16] asked whether there exists a generalization of `Eig` to k-uniform hypergraphs which is equivalent to `Disc`. The eigenvalue description of graph quasirandomness has proved to be a very useful result to show that certain explicitly constructed graphs are quasirandom (see [3, 4, 33, 41]).

This leads to our first main contribution. We define a generalization of `Eig` to k-uniform hypergraphs and add it into the equivalences stated in Theorem 1. This answers the aforementioned question of Conlon et al. [16].

Our second contribution is to generalize Theorem 1 to a slightly larger class of hypergraphs. Let $k \geq 2$ be an integer and let π be a proper partition of k, by which we mean that π is an unordered list of at least two positive integers whose sum is k. For the partition π of k given by $k = k_1 + \cdots + k_t$, we will abuse notation by saying that $\pi = k_1 + \cdots + k_t$. For every proper partition π, we define properties `Expand`$[\pi]$, `Eig`$[\pi]$, and `Cycle`$_4[\pi]$ and show that they are equivalent.

Definition. *Let $k \geq 2$ and let $\pi = k_1 + \cdots + k_t$ be a proper partition of k. A k-uniform hypergraph F is π-linear if there exists an ordering $E_1, \ldots, E_m$ of the edges of F such that for every i, there exists a partition of the vertices of E_i into $A_{i,1}, \ldots, A_{i,t}$ such that for $1 \leq s \leq t$, $|A_{i,s}| = k_s$ and for every $j < i$, there exists an s such that $E_j \cap E_i \subseteq A_{i,s}$.*

Our hypergraph eigenvalues are based on definitions of Friedman and Wigderson [21, 22] (see Section 3). In graphs, it is easier to study the eigenvalues of regular graphs (possibly with loops). A similar situation occurs for hypergraphs, so Friedman and Wigderson [21, 22] focused almost exclusively on the following notion of regular for hypergraphs.

Definition. *A k-uniform hypergraph with loops H consists of a finite set $V(H)$ and a collection $E(H)$ of k-element multisets of elements from $V(H)$. Informally, every edge has size exactly k but a vertex is allowed to be repeated inside of an edge. A k-uniform hypergraph with loops H is d-coregular if for every $(k-1)$-multiset S, there are exactly d edges which contain S.*

The following is our main theorem.

Theorem 2. (Main Result) *Let $0 < p < 1$ be a fixed constant and let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of k-uniform hypergraphs with loops such that $|V(H_n)| = n$ and H_n is $\lfloor pn \rfloor$-coregular. Let $\pi = k_1 + \cdots + k_t$ be a proper partition of k. The following properties are equivalent:*

- **$Eig[\pi]$:** $\lambda_{1,\pi}(H_n) = pn^{k/2} + o(n^{k/2})$ *and* $\lambda_{2,\pi}(H_n) = o(n^{k/2})$, *where $\lambda_{1,\pi}(H_n)$ and $\lambda_{2,\pi}(H_n)$ are the first and second largest eigenvalues of H_n with respect to π, which are defined in Section 3.*

- **$Expand[\pi]$:** *For all $S_i \subseteq \binom{V(H_n)}{k_i}$ where $1 \leq i \leq t$,*

$$e(S_1,\ldots,S_t) = p \prod_{i=1}^{t} |S_i| + o\left(n^k\right)$$

 where $e(S_1,\ldots,S_t)$ is the number of tuples $(s_1,\ldots,s_t)$ such that $s_1 \cup \cdots \cup s_t$ is a hyperedge and $s_i \in S_i$.

- **$Count[\pi\text{-linear}]$:** *If F is an f-vertex, m-edge, k-uniform, π-linear hypergraph, then the number of labeled copies of F in H_n is $p^m n^f + o(n^f)$.*

- **$Cycle_4[\pi]$:** *The number of labeled copies of $C_{\pi,4}$ in H_n is at most $p^{|E(C_{\pi,4})|} n^{|V(C_{\pi,4})|} + o(n^{|V(C_{\pi,4})|})$, where $C_{\pi,4}$ is the hypergraph four cycle of type π which is defined in Section 2.*

- **$Cycle_{4\ell}[\pi]$:** *the number of labeled copies of $C_{\pi,4\ell}$ in H_n is at most $p^{|E(C_{\pi,4\ell})|} n^{|V(C_{\pi,4\ell})|} + o(n^{|V(C_{\pi,4\ell})|})$, where $C_{\pi,4\ell}$ is the hypergraph cycle of type π and length 4ℓ defined in Section 2.*

In fact, all implications above except $Cycle_{4\ell}[\pi] \Rightarrow Eig[\pi]$ are true with the coregular condition replaced by the weaker condition that $|E(H_n)| \geq p\binom{n}{k} + o(n^k)$.

Remarks.

- In a companion paper [32], we prove that $Cycle_{4\ell}[\pi] \Rightarrow Eig[\pi]$ for all sequences $\mathcal{H} = \{H_n\}_{n\to\infty}$ where H_n is a k-uniform hypergraph with loops, $|V(H_n)| = n$, and $|E(H_n)| \geq p\binom{n}{k} + o(n^k)$.

- Following Chung, Graham, and Wilson [15], our results extend to sequences which are not defined for every n as follows. Let $\mathcal{H} = \{H_{n_q}\}_{q\to\infty}$ be a sequence of hypergraphs such that $|V(H_{n_q})| = n_q$, $n_q < n_{q+1}$, and $|E(H_{n_q})| \geq p\binom{n_q}{k} + o(n_q^k)$, where now the little-$o$ expression means there exists a function $f(q)$ such that $|E(H_{n_q})| \geq p\binom{n_q}{k} + f(q)$ with $\lim_{q\to\infty} f(q)n_q^{-k} = 0$. Similarly, when we say that property P (which might include a little-o expression) implies a property P', what we mean is that there exist functions $f(q)$ and $f'(q)$ such that $P(f(q))$ implies $P'(f'(q))$, where the notation $P(f(q))$ stands for the property P with the little-o replaced by the function $f(q)$.

- If $\pi = 1 + \cdots + 1$, the partition of k into k ones, then the equivalences $\mathtt{Expand}[\pi] \Leftrightarrow \mathtt{Count}[\pi\text{-linear}] \Leftrightarrow \mathtt{Cycle}_4[\pi]$ of Theorem 2 constitute Theorem 1. Therefore, the property $\mathtt{Eig}[1 + \cdots + 1]$ is the spectral property that is equivalent to the weak quasirandom properties studied by Kohayakawa, Rödl, and Skokan [29] and Conlon, Hàn, Person, and Schacht [16].

- If π' is a refinement of π, then clearly $\mathtt{Count}[\pi\text{-linear}] \Rightarrow \mathtt{Count}[\pi'\text{-linear}]$ and so if $\{H_n\}_{n\to\infty}$ is a sequence satisfying the properties in Theorem 2 for π, it satisfies the properties for π'. In a companion paper [34], we show the converse: if π' is not a refinement of π then $\mathtt{Expand}[\pi] \not\Rightarrow \mathtt{Expand}[\pi']$ so the property $\mathtt{Expand}[\pi]$ is distinct for distinct π and arranged in a poset via partition refinement.

The remainder of this paper is organized as follows. In Section 2, we define the hypergraph cycles $C_{\pi,4}$. Section 3 gives the formal definition of eigenvalues with respect to π. Theorem 2 is proved by showing a chain of implications in the order stated in the theorem; Section 4 proves $\mathtt{Eig}[\pi] \Rightarrow \mathtt{Expand}[\pi]$, Section 5 proves $\mathtt{Expand}[\pi] \Rightarrow \mathtt{Count}[\pi\text{-linear}]$, and Section 6 shows that $\mathtt{Cycle}_{4\ell}[\pi] \Rightarrow \mathtt{Eig}[\pi]$ for d-coregular hypergraphs with loops. Throughout this paper, we use the notation $[n] = \{1, \ldots, n\}$.

2 Hypergraph Cycles

In this section, we define the hypergraph cycles $C_{\pi,2\ell}$. The hypergraph cycles $C_{\pi,2\ell}$ are defined by first defining steps, then defining a path as a combination of steps, and finally defining the cycle as a path with its endpoints identified.

Definition. *Let $\vec{\pi} = (1, \ldots, 1)$ be the ordered partition of t into t parts. Define the* step *of type $\vec{\pi}$, denoted $S_{\vec{\pi}}$, as follows. Let A be a vertex set of size 2^{t-1} where elements are labeled by binary strings of length $t - 1$ and let $B_2, \ldots, B_t$ be disjoint sets of size 2^{t-2} where elements are labeled by binary strings of length $t - 2$. The vertex set of $S_{\vec{\pi}}$ is the disjoint union $A \dot\cup B_2 \dot\cup \cdots \dot\cup B_t$. Make $\{a, b_2, \ldots, b_t\}$ a hyperedge of $S_{\vec{\pi}}$ if $a \in A$, $b_j \in B_j$, and the code for b_{j+1} is equal to the code formed by removing the jth bit of the code for a.*

For a general $\vec{\pi} = (k_1, \ldots, k_t)$, start with $S_{(1,\ldots,1)}$ and enlarge each vertex into the appropriate size; that is, a vertex in A is expanded into k_1 vertices and each vertex in B_j is expanded into k_j vertices. More precisely, the vertex set of $S_{\vec{\pi}}$ is $(A \times [k_1]) \dot\cup (B_2 \times [k_2]) \dot\cup \cdots \dot\cup (B_t \times [k_t])$, and if $\{a, b_2, \ldots, b_t\}$ is an edge of $S_{(1,\ldots,1)}$, then $\{(a, 1), \ldots, (a, k_1), (b_2, 1), \ldots, (b_2, k_2), \ldots, (b_t, 1), \ldots, (b_t, k_t)\}$ is a hyperedge of $S_{\vec{\pi}}$.

This defines the step *of type $\vec{\pi}$, denoted $S_{\vec{\pi}}$. Let $A^{(0)}$ be the ordered tuple of vertices of A in $S_{\vec{\pi}}$ whose binary code ends with zero and $A^{(1)}$ the ordered tuple of vertices of A whose binary code ends with one, where vertices are listed in lexicographic order within each $A^{(i)}$. These tuples $A^{(0)}$ and $A^{(1)}$ are the two* attach tuples *of $S_{\vec{\pi}}$*

Figure 1 shows the steps of type $(1, 1)$ and type $(3, 2)$. Notice that each step has "length" two if we consider the attach tuples as the "ends" of a path.
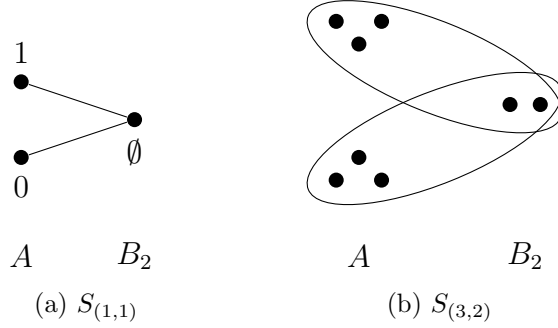
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(a) $S_{(1,1)}$ (b) $S_{(3,2)}$

Figure 1: Steps with $t = 2$

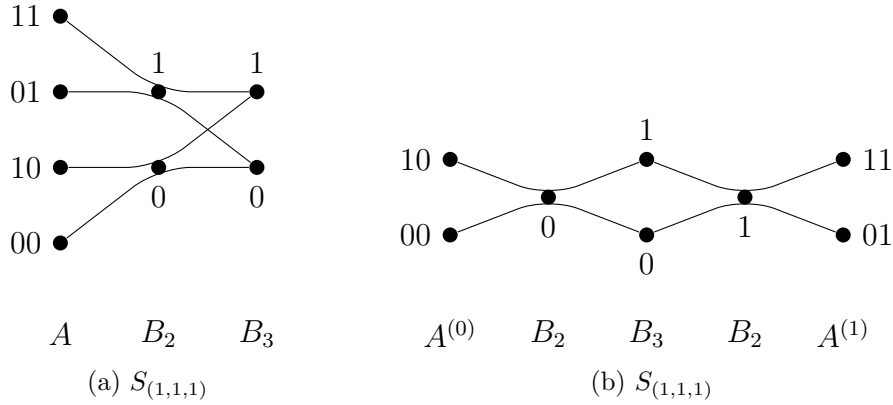(a) $S_{(1,1,1)}$ (b) $S_{(1,1,1)}$

Figure 2: Steps of type $\pi = (1, 1, 1)$

Figure 2 shows two different drawings of the step of type $\vec{\pi} = (1, 1, 1)$. Notice that the attach tuples are easily visible in Figure 2 *(b)*, since the two attach tuples are the codes in A ending with a zero and a one. The step of type $\vec{\pi} = (k_1, k_2, k_3)$ is an enlarged version of Figure 2 similar to Figure 1 *(b)*.

In general for arbitrary $\vec{\pi}$, the step $S_{\vec{\pi}}$ can be drawn in two ways similar to Figure 2. First from the definition, a step is a k-partite hypergraph with parts $A, B_2, \ldots, B_t$ so it can be drawn similar to Figure 2 *(a)*. But the step can also be drawn with the two attach tuples on separate ends of the picture like Figure 2 *(b)*. Let M_0 be the set of edges incident to vertices in the attach tuple $A^{(0)}$ and M_1 the set of edges incident to vertices in $A^{(1)}$. Edges from M_0 and M_1 intersect only in vertices in B_t because if $a_0 \in A^{(0)}$ and $a_1 \in A^{(1)}$ then the code for a_0 ends in a zero and the code for a_1 ends in a one, so only when deleting the last bit will the codes possibly be the same. Therefore, the step $S_{\vec{\pi}}$ can be viewed as a type of length two path in a hypergraph formed from a collection of k-partite edges M_0 between $A^{(0)}$ and B_t and another collection of k-partite edges M_1 between B_t and $A^{(1)}$.

Definition. *Let $\ell \geq 1$. The* path of type $\vec{\pi}$ of length 2ℓ, *denoted $P_{\vec{\pi}, 2\ell}$, is the hypergraph formed from ℓ copies of $S_{\vec{\pi}}$ with successive attach tuples identified. That is, let $T_1, \ldots, T_\ell$*

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be copies of $S_{\vec{\pi}}$ and let $A_i^{(0)}$ and $A_i^{(1)}$ be the attach tuples of T_i. The hypergraph $P_{\vec{\pi},2\ell}$ is the hypergraph consisting of $T_1, \ldots, T_\ell$ where the vertices of $A_i^{(1)}$ are identified with $A_{i+1}^{(0)}$ for every $1 \leq i \leq \ell - 1$. (Recall that by definition, $A_i^{(1)}$ and $A_{i+1}^{(0)}$ are tuples (i.e. ordered lists) of vertices, so the identification of $A_i^{(1)}$ and $A_{i+1}^{(0)}$ identifies the corresponding vertices in these tuples.) The attach tuples of $P_{\vec{\pi},2\ell}$ are the tuples $A_1^{(0)}$ and $A_\ell^{(1)}$.
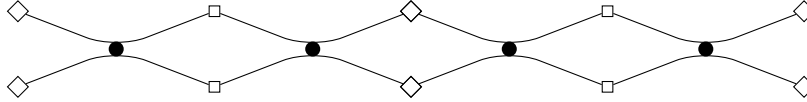

Figure 3: $P_{(1,1,1),4}$

In Figure 3, the path $P_{(1,1,1),4}$ is drawn as two copies of $S_{(1,1,1)}$ with attach tuples identified. The diamond, circle, and square vertices keep track of the parts A, B_2, B_3. For a general $P_{(k_1,k_2,k_3),4}$, each diamond vertex is enlarged into k_1 vertices, each circle vertex is enlarged into k_2 vertices, and each square vertex is enlarged into k_3 vertices. For a general $\vec{\pi}$, every step can be visualised as in Figure 2 *(b)* as two collections of k-partite edges M_0 and M_1 between A and B_t, so all paths $P_{\vec{\pi},2\ell}$ can be visualised as in Figure 3 as a concatenation of steps.

Definition. *Let $\ell \geq 2$. The* cycle of type π and length 2ℓ, *denoted $C_{\pi,2\ell}$, is the hypergraph formed by picking any ordering $\vec{\pi}$ of π and identifying the attach tuples of $P_{\vec{\pi},2\ell}$.*

The definition of $C_{\pi,2\ell}$ is independent of the ordering $\vec{\pi}$; a proof appears in [31].

Definition. *Let $\ell \geq 2$. A* walk of type $\vec{\pi}$ and length 2ℓ *in a hypergraph H is a function $f : V(P_{\vec{\pi},2\ell}) \to V(H)$ that preserves edges. Informally, a walk is a path where the vertices are not necessarily distinct. A* circuit of type π of length 2ℓ *in a hypergraph H is a function $f : V(C_{\pi,2\ell}) \to V(H)$ that preserves edges. Informally, a circuit is a cycle where the vertices are not necessarily distinct.*

There are two alternative definitions of the cycle of length four. First, Conlon et al. [16] defined a cycle of length four for $\pi = 1 + \cdots + 1$ by an operation called *reflection*. Our definition of $C_{1+\cdots+1,4}$ is equivalent to the definition in [16]; this can be seen by noticing that the bit strings in our definition keep track of the vertex duplications which occur during reflection.

Finally, there is a concise direct definition of the cycle of type π and length four which avoids the complexity of defining steps and paths. We will not use this shorter definition in this paper, instead working with steps, paths, and walks, but we include this short definition for completeness. Let $D_1, \ldots, D_t$ be disjoint sets of size 2^{t-1} whose elements are labeled by $(t-1)$-length binary strings. The vertex set is $D_1 \dot{\cup} \ldots \dot{\cup} D_t$. For $d_1 \in D_1, \ldots, d_t \in D_t$, make $\{d_1, \ldots, d_t\}$ a hyperedge if there exists a binary string s of length t such that the code for d_i equals the code formed by deleting the ith bit of s. The cycle for general π is formed by enlarging this cycle appropriately. Figure 4 shows cycles drawn using this definition.
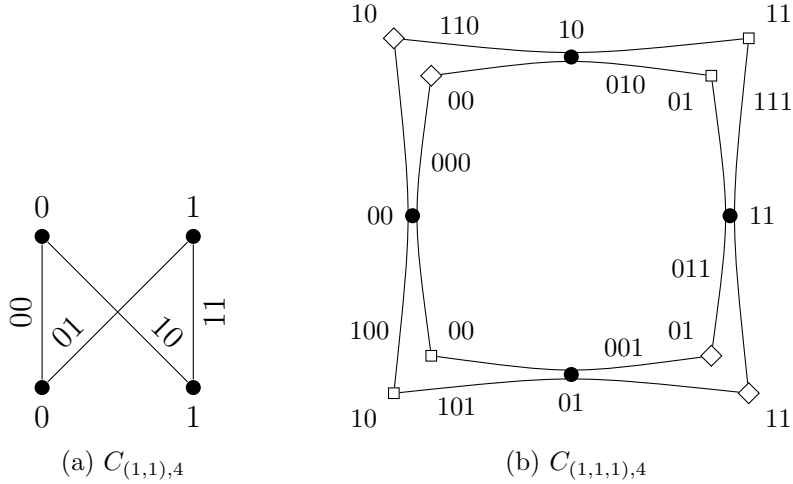
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Figure 4: Alternate definition of the cycle of length four

3 Hypergraph Eigenvalues

This section contains the definition of the largest and second largest eigenvalues of a hypergraph with respect to π and also contains some discussion and basic facts about them.

There have been three independently developed approaches to hypergraph eigenvalues: a definition by Chung [11] and Lu and Peng [35, 36] using matrices, an approach of Friedman and Wigderson [21, 22] and Cooper and Dutle [17], and lastly the eigenvalues of the shadow graph [7, 19, 37, 38, 40]. The definitions of Friedman and Wigderson [21, 22] are most suitable for our purposes and we will use their definitions as our starting point.

Definition. *Let $V_1, \ldots, V_k$ be finite-dimensional vector spaces over $\mathbb{R}$. A k-linear map is a function $\phi : V_1 \times \cdots \times V_k \to \mathbb{R}$ such that for each $1 \leq i \leq k$, ϕ is linear in the ith coordinate. That is, for every fixed $x_i \in V_i$, $\phi(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$ is a linear map from V_i to $\mathbb{R}$. A k-linear map $\phi : V^k \to \mathbb{R}$ is* symmetric *if for all permutations η of $[k]$ and all $x_1, \ldots, x_k \in V$, $\phi(x_1, \ldots, x_k) = \phi(x_{\eta(1)}, \ldots, x_{\eta(k)})$.*

Definition. *Let $V_1, \ldots, V_k$ be finite-dimensional vector spaces over $\mathbb{R}$, let $B_i = \{b_{i,1}, \ldots, b_{i,\dim(V_i)}\}$ be an orthonormal basis of V_i, and let $\phi : B_1 \times \cdots \times B_k \to \mathbb{R}$ be any map. Extending ϕ* linearly *to $V_1 \times \cdots \times V_k$ means that ϕ is extended to a map $V_1 \times \cdots \times V_k \to \mathbb{R}$ where for $x_1 \in V_1, \ldots, x_k \in V_k$,*

$$\phi(x_1, \ldots, x_k) = \sum_{j_1=1}^{\dim(V_1)} \cdots \sum_{j_k=1}^{\dim(V_k)} \langle x_1, b_{1,j_1} \rangle \cdots \langle x_k, b_{k,j_k} \rangle \phi(b_{1,j_1}, \ldots, b_{k,j_k}). \tag{1}$$

Note that extending ϕ in this way produces a k-linear map.

Definition. *(Friedman and Wigderson [21, 22]) Let H be a k-uniform hypergraph with loops. The* adjacency map *of H is the symmetric k-linear map $\tau_H : W^k \to \mathbb{R}$ defined as*

follows, where W is the vector space over $\mathbb{R}$ of dimension $|V(H)|$. First, for all $v_1, \ldots, v_k \in V(H)$, let

$$\tau_H(e_{v_1}, \ldots, e_{v_k}) = \begin{cases} 1 & \{v_1, \ldots, v_k\} \in E(H), \\ 0 & otherwise, \end{cases}$$

where e_v denotes the indicator vector of the vertex v, that is the vector which has a one in coordinate v and zero in all other coordinates. We have defined the value of τ_H when the inputs are standard basis vectors of W. Extend τ_H to all the domain linearly.

Definition. *Let $W_1, \ldots, W_k$ be finite dimensional vector spaces over $\mathbb{R}$, let $\|\cdot\|$ denote the Euclidean 2-norm on W_i, and let $\phi : W_1 \times \cdots \times W_k \to \mathbb{R}$ be a k-linear map. The* spectral norm *of ϕ is*

$$\|\phi\| = \sup_{\substack{x_i \in W_i \\ \|x_i\|=1}} |\phi(x_1, \ldots, x_k)| .$$

Before defining the first and second largest eigenvalue of H with respect to a general partition π, we give the definitions when $\pi = 1 + \cdots + 1$, that is π is the partition into k ones.

Definition. *Let H be an n-vertex, k-uniform hypergraph, let W be the vector space over $\mathbb{R}$ of dimension n, and let $J : W^k \to \mathbb{R}$ be the all-ones map. That is, if $e_{i_1}, \ldots, e_{i_k}$ are any standard basis vectors of W, then $J(e_{i_1}, \ldots, e_{i_k}) = 1$, and J is extended linearly to all of the domain as in* (1).

The largest eigenvalue of H with respect to $\pi = 1 + \cdots + 1$ is $\|\tau_H\|$ *and the* second largest eigenvalue of H with respect to $\pi = 1 + \cdots + 1$ is $\left\|\tau_H - \frac{k!|E(H)|}{n^k} J\right\|$.

In order to extend this definition to general $\vec{\pi} = (k_1, \ldots, k_t)$, it is convenient to use the language of tensor products.

Definition. *Let V and W be finite dimensional vector spaces over $\mathbb{R}$ of dimension n and m respectively. The* tensor product *of V and W, written $V \otimes W$, is the vector space over $\mathbb{R}$ of dimension nm. A typical tensor a in $V \otimes W$ has the form $a = \sum_{i=1}^{\dim(V)} \sum_{j=1}^{\dim(W)} \alpha_{i,j}(e_i \otimes e'_j)$, where $\alpha_{i,j} \in \mathbb{R}$ and $e_1, \ldots e_{\dim(V)}$ is the standard basis of V and $e'_1, \ldots, e'_{\dim(W)}$ is the standard basis of W. The* length *of a tensor is the length of the vector in the vector space $V \otimes W$. Thus the length of a is $\left(\sum_{i=1}^{\dim(V)} \sum_{j=1}^{\dim(W)} \alpha_{i,j}^2\right)^{1/2}$.*

We are now ready to define the map $\tau_{\vec{\pi}}$ and then the first and second largest eigenvalue of H with respect to π for a general π. In the definition, think of the tensor product $W^{\otimes k_i}$ as a vector space of dimension $|V(H)|^{k_i}$ indexed by ordered k_i-sets of vertices.

Definition. *Let W be a finite dimensional vector space over $\mathbb{R}$, let $\sigma : W^k \to \mathbb{R}$ be any k-linear function, and let $\vec{\pi}$ be a proper ordered partition of k, so $\vec{\pi} = (k_1, \ldots, k_t)$ for some*

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integers $k_1, \ldots, k_t$ *with* $t \geq 2$. *Now define a t-linear function* $\sigma_{\vec{\pi}} : W^{\otimes k_1} \times \cdots \times W^{\otimes k_t} \to \mathbb{R}$ *by first defining* $\sigma_{\vec{\pi}}$ *when the inputs are basis vectors of* $W^{\otimes k_i}$ *and then extending linearly. For each* i, $B_i = \{b_{i,1} \otimes \cdots \otimes b_{i,k_i} : b_{i,j}$ *is a standard basis vector of* $W\}$ *is a basis of* $W^{\otimes k_i}$, *so for each* i, *pick* $b_{i,1} \otimes \cdots \otimes b_{i,k_i} \in B_i$ *and define*

$$\sigma_{\vec{\pi}}(b_{1,1} \otimes \cdots \otimes b_{1,k_1}, \ldots, b_{t,1} \otimes \cdots \otimes b_{t,k_t}) = \sigma(b_{1,1}, \ldots, b_{1,k_1}, \ldots, b_{t,1}, \ldots, b_{t,k_t}).$$

Now extend $\sigma_{\vec{\pi}}$ *linearly to all of the domain.* $\sigma_{\vec{\pi}}$ *will be t-linear since* σ *is k-linear.*

Definition. *Let* H *be a k-uniform hypergraph with loops and let* $\tau = \tau_H$ *be the (k-linear) adjacency map of* H. *Let* π *be any (unordered) partition of* k *and let* $\vec{\pi}$ *be any ordering of* π. *The* largest *and* second largest *eigenvalues of* H *with respect to* π, *denoted* $\lambda_{1,\pi}(H)$ *and* $\lambda_{2,\pi}(H)$, *are defined as*

$$\lambda_{1,\pi}(H) := \|\tau_{\vec{\pi}}\| \quad and \quad \lambda_{2,\pi}(H) := \left\| \tau_{\vec{\pi}} - \frac{k!|E(H)|}{n^k} J_{\vec{\pi}} \right\|.$$

Both $\lambda_{1,\pi}(H)$ and $\lambda_{2,\pi}(H)$ are well defined since for any two orderings $\vec{\pi}$ and $\vec{\pi}'$ of π, $\tau_{\vec{\pi}} = \tau_{\vec{\pi}'}$ and $J_{\vec{\pi}} = J_{\vec{\pi}'}$ since both τ and J are symmetric maps.

Remarks.

- For a graph G ($k = 2$ and $\pi = 1+1$), $\lambda_{1,1+1}(G)$ equals the largest eigenvalue in absolute value of the adjacency matrix A of G since both are equal to $\sup\{|x^T A x| : \|x\| = 1\}$. Additionally, if G is d-regular, then $\lambda_{2,1+1}(G)$ equals the second largest eigenvalue of A in absolute value. Indeed, if G is a d-regular graph, then $2|E(H)|/n^2 = \frac{d}{n}$, so $\lambda_{2,1+1}(G) = \left\| \tau_G - \frac{d}{n} J \right\|$. The bilinear map $\tau_G - \frac{d}{n} J$ corresponds to the matrix $A - \frac{d}{n} J$ where J is now the all-ones matrix. The largest eigenvalue of $A - \frac{d}{n} J$ in absolute value is the second largest eigenvalue of A in absolute value, and this equals the spectral norm of the respective map.

- For any k-uniform hypergraph H, $\lambda_{1,1+\cdots+1}(H)$ exactly matches the definition of Friedman and Wigderson [21, 22]. [21, 22] did not define the second largest eigenvalue for all hypergraphs. For d-coregular hypergraphs with loops, [21, 22] defined the second largest eigenvalue and it exactly corresponds to our definition of $\lambda_{2,1+\cdots+1}(H)$, where $\frac{k!|E(H)|}{n^k} = \frac{d}{n}$ (recall that H has loops which is why n^k appears in the denominator instead of the falling factorial). For the random hypergraph $G^{(k)}(n,p)$, [21, 22] also defined a second largest eigenvalue with respect to density p as the spectral norm of $\tau_{G(n,p)} - pJ$. While different than our definition, $p = (1 + o(1))\frac{k!|E(G(n,p))|}{n^k}$ so the definitions are similar.

- If H is a k-uniform, d-coregular hypergraph with loops, Friedman and Wigderson [21, 22] proved several facts about $\lambda_{1,1+\cdots+1}(H)$ and $\lambda_{2,1+\cdots+1}(H)$. First, $\lambda_{1,1+\cdots+1}(H) = dn^{(k-2)/2}$ and the supremum is achieved by the all-ones vectors scaled to unit length. They also proved several facts about $\lambda_{2,1+\cdots+1}(H)$ including upper and lower bounds, an Expander Mixing Lemma which we generalize to all π in Theorem 3, and the asymptotic value of $\lambda_{2,1+\cdots+1}(G(n,p))$.

4 Eig[π] ⇒ Expand[π]

In this section we prove a generalization of the graph Expander Mixing Lemma which relates spectral and expansion properties of graphs. The graph version was first discovered independently by Alon and Milman [2] and Tanner [42]. For background on graph expansion and eigenvalues, see [1, 5, 26]. The following theorem extends the hypergraph Expander Mixing Lemma of Friedman and Widgerson [21, 22], which applied for $\pi = 1 + \cdots + 1$. The theorem is stated for ordered partitions $\vec{\pi}$, but trivially gives the same result for any ordering $\vec{\pi}$ of a partition π.

Theorem 3. *(Hypergraph Expander Mixing Lemma) Let H be an n-vertex, k-uniform hypergraph with loops. Let $\vec{\pi} = (k_1, \ldots, k_t)$ be a proper ordered partition of k and let $S_i \subseteq \binom{V(H)}{k_i}$ for $1 \le i \le t$ (where the elements of S_i are potentially multisets of size k_i). Then*

$$\left| e(S_1, \ldots, S_t) - \frac{k! |E(H)|}{n^k} \prod_{i=1}^{t} |S_i| \right| \le \lambda_{2,\pi}(H) \sqrt{|S_1| \cdots |S_t|},$$

where $e(S_1, \ldots, S_t)$ is the number of ordered tuples $(s_1, \ldots, s_t)$ such that $s_1 \cup \cdots \cup s_t \in E(H)$ and $s_i \in S_i$.

Proof. Let $q = \frac{k! |E(H)|}{n^k}$, let τ_H be the adjacency map of H, and let $\sigma = \tau_H - qJ$. It is easy to see that by definition, $(\tau - qJ)_{\vec{\pi}} = \tau_{\vec{\pi}} - qJ_{\vec{\pi}}$, so $\lambda_{2,\pi}(H) = \|\sigma_{\vec{\pi}}\|$. Let $\chi_{S_i} \in W^{k_i \otimes}$ be the indicator tensor of S_i. If we let $V(H) = [n]$, then

$$\chi_{S_i} = \sum_{\substack{\{v_1, \ldots, v_{k_i}\} \in S_i \\ v_1 \le \cdots \le v_{k_i}}} (e_{v_1} \otimes \cdots \otimes e_{v_{k_i}}).$$

By the linearity of $\sigma_{\vec{\pi}}$ and the definition of $J_{\vec{\pi}}$,

$$\sigma_{\vec{\pi}}(\chi_{S_1}, \ldots, \chi_{S_t}) = \tau_{\vec{\pi}}(\chi_{S_1}, \ldots, \chi_{S_t}) - qJ_{\vec{\pi}}(\chi_{S_1}, \ldots, \chi_{S_t}) = e(S_1, \ldots, S_t) - q \prod_{i=1}^{t} |S_i|.$$

Before upper bounding this by $\lambda_{2,\pi}(H)$, we must scale each indicator tensor to be unit length. Since $\{e_{j_1} \otimes \cdots \otimes e_{j_{k_i}} : 1 \le j_1, \ldots, j_{k_i} \le n\}$ forms a basis of $W^{\otimes k_i}$, we have $\|\chi_{S_i}\| = \sqrt{|S_i|}$. Thus

$$\left| \sigma_{\vec{\pi}} \left(\frac{\chi_{S_1}}{\|\chi_{S_1}\|}, \ldots, \frac{\chi_{S_t}}{\|\chi_{S_t}\|} \right) \right| \le \|\sigma_{\vec{\pi}}\| = \lambda_{2,\pi}(H).$$

Consequently,

$$|\sigma_{\vec{\pi}}(\chi_{S_1}, \ldots, \chi_{S_t})| \le \lambda_{2,\pi}(H) \|\chi_{S_1}\| \cdots \|\chi_{S_t}\| = \lambda_{2,\pi}(H) \sqrt{|S_1| \cdots |S_t|},$$

and the proof is complete. □

Lemma 4. *Let $\mathcal{H} = \{H_n\}$ be a sequence of k-uniform hypergraphs with loops with $|V(H_n)| = n$ and $|E(H_n)| \geq p\binom{n}{k} + o(n^k)$. Let τ_n be the adjacency map of H_n and let $\vec{\pi} = (k_1, \ldots, k_t)$ be a proper ordered partition of k. If $\lambda_{1,\pi}(H_n) = pn^{k/2} + o(n^{k/2})$, then $|E(H_n)| = p\binom{n}{k} + o(n^k)$.*

Proof. Throughout this proof the subscripts on n are dropped for simplicity. Let W be the vector space over $\mathbb{R}$ of dimension n. For $1 \leq i \leq t$, let $\vec{1}_{k_i}$ denote the all-ones vector in $W^{\otimes k_i}$, so $\|\vec{1}_{k_i}\| = n^{k_i/2}$. Then

$$\tau_{\vec{\pi}}\left(\frac{\vec{1}_{k_1}}{n^{k_1/2}}, \ldots, \frac{\vec{1}_{k_t}}{n^{k_t/2}}\right) = \frac{1}{n^{k/2}}\tau_{\vec{\pi}}\left(\vec{1}_{k_1}, \ldots, \vec{1}_{k_t}\right) = \frac{1}{n^{k/2}}\tau(\vec{1}_1, \ldots, \vec{1}_1)$$

$$= \frac{1}{n^{k/2}}\sum_{i_1,\ldots,i_k=1}^{n}\tau(e_{i_1}, \ldots, e_{i_k})$$

$$= \frac{1}{n^{k/2}}k!|E(H)|.$$

Thus the spectral norm of $\tau_{\vec{\pi}}$ is at least $k!|E(H)|/n^{k/2}$, so

$$pn^{k/2} \leq \frac{k!|E(H)|}{n^{k/2}} + o(n^{k/2}) \leq \|\tau_{\vec{\pi}}\| + o(n^{k/2}) = pn^{k/2} + o(n^{k/2})$$

This implies equality (up to $o(n^{k/2})$) throughout the above expression. In particular, $|E(H_n)| = p\binom{n}{k} + o(n^k)$. $\square$

Proof that $\mathtt{Eig}[\pi] \Rightarrow \mathtt{Expand}[\pi]$. First, $\mathtt{Eig}[\pi]$ contains the assertion that $\lambda_{1,\pi}(H_n) = pn^{k/2} + o(n^{k/2})$ which by Lemma 4 implies $|E(H_n)| = p\binom{n}{k} + o(n^k)$. Consequently, $k!|E(H_n)|/n^k = (1 + o(1))p$ and Theorem 3 imply that

$$\left|e(S_1, \ldots, S_t) - (1 + o(1))p\prod_{i=1}^{t}|S_i|\right| \leq \lambda_{2,\pi}(H)\sqrt{|S_1|\cdots|S_t|} \tag{2}$$

for any choice of $S_i \subseteq \binom{V(H_n)}{k_i}$, $i = 1, \ldots, t$. Since π is a partition of k, $\sqrt{|S_1|\cdots|S_t|} = O(n^{k/2})$. Also, $\mathtt{Eig}[\pi]$ states that $\lambda_{2,\pi}(H) = o(n^{k/2})$. Thus (2) becomes

$$\left|e(S_1, \ldots, S_t) - p|S_1|\cdots|S_t|\right| = o(n^k),$$

which proves $\mathtt{Expand}[\pi]$. $\square$

5 $\mathtt{Expand}[\pi] \Rightarrow \mathtt{Count}[\pi\text{-linear}]$

The proof that $\mathtt{Expand}[\pi] \Rightarrow \mathtt{Count}[\pi\text{-linear}]$ follows from an embedding lemma for hypergraphs. The proof of Proposition 5 below is a generalization of an argument by Kohayakawa et al. [28] who proved it in the special case of linear hypergraphs, so we omit the proof. A detailed proof appears online [31]. The proposition below is stated for ordered partitions $\vec{\pi}$, but it is easy to see that the proposition is independent of the ordering chosen for $\vec{\pi}$.

Proposition 5. *Let $\vec{\pi} = (k_1, \ldots, k_t)$ be a proper ordered partition of k, let $0 < p < 1$, and let F be any fixed k-uniform, π-linear hypergraph with f vertices and m edges.*

Let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of k-uniform hypergraphs with loops with $|V(H_n)| = n$, $|E(H_n)| = p\binom{n}{k} + o(n^k)$, and for which $\mathtt{Expand}[\vec{\pi}]$ holds. In other words, for every $S_1 \subseteq \binom{V(H)}{k_1}, \ldots, S_t \subseteq \binom{V(H)}{k_t}$, we have $e(S_1, \ldots, S_t) = p|S_1| \cdots |S_t| + o(n^k)$. Then the number of labeled copies of F in H is $p^m n^f + o(n^f)$.

6 $\mathtt{Cycle}_{4\ell}[\pi] \Rightarrow \mathtt{Eig}[\pi]$

In this section, we prove that if $\mathcal{H}$ is a sequence of d-coregular, k-uniform hypergraphs with loops which satisfies $\mathtt{Cycle}_{4\ell}[\pi]$, then $\mathcal{H}$ satisfies $\mathtt{Eig}[\pi]$. Indeed, if H is d-coregular with loops, then $\lambda_{1,\pi}(H) = dn^{k/2-1}$ and the vectors maximizing $\tau_{\vec{\pi}}$ are the all-ones vectors scaled to unit length (see [21, 22]). These facts simplify the proof of $\mathtt{Cycle}_{4\ell}[\pi] \Rightarrow \mathtt{Eig}[\pi]$ which appears in this section. In a companion paper [32], we develop the additional algebra required to prove $\mathtt{Cycle}_{4\ell}[\pi] \Rightarrow \mathtt{Eig}[\pi]$ for all sequences. Throughout this section, let $0 < p < 1$ be a fixed integer and define $d = d(n) = \lfloor pn \rfloor$.

First, let us recall the proof of $\mathtt{Cycle}_4[1+1] \Rightarrow \mathtt{Eig}[1+1]$ for graphs. Let A be the adjacency matrix of a d-regular graph G. Then $\mathrm{Tr}\,[A^4]$ is the number of circuits of length 4 so $\mathtt{Cycle}_4[1+1]$ implies that $\mathrm{Tr}\,[A^4] = d^4 + o(n^4)$. Since G is d-regular, the largest eigenvalue of A^4 is d^4 so that all eigenvalues of A besides d are $o(n)$ in absolute value, completing the proof that $\mathtt{Eig}[1+1]$ holds. Our proof for hypergraphs follows the same outline once some algebraic facts about multilinear maps are proved. In Section 6.1, we define (non-standard) products and powers of multilinear maps. In Section 6.2, we show that the powers of multilinear maps count walks and that the trace of the powers of multilinear maps counts circuits. Finally, Section 6.3 contains the proof that $\mathtt{Cycle}_{4\ell}[\pi] \Rightarrow \mathtt{Eig}[\pi]$.

6.1 Products and powers of multilinear maps

In this section, we give (non-standard) definitions of the products and powers of multilinear maps.

Definition. *Let $V_1, \ldots, V_t$ be finite dimensional vector spaces over $\mathbb{R}$ and let $\phi, \psi : V_1 \times \cdots \times V_t \to \mathbb{R}$ be t-linear maps. The* product *of ϕ and ψ, written $\phi * \psi$, is a $(t-1)$-linear map defined as follows. Let $u_1, \ldots, u_{t-1}$ be vectors where $u_i \in V_i$. Let $\{b_1, \ldots, b_{\dim(V_t)}\}$ be any orthonormal basis of V_t.*

$$\phi * \psi : (V_1 \otimes V_1) \times (V_2 \otimes V_2) \times \cdots \times (V_{t-1} \otimes V_{t-1}) \to \mathbb{R}$$

$$\phi * \psi(u_1 \otimes v_1, \ldots, u_{t-1} \otimes v_{t-1}) := \sum_{j=1}^{\dim(V_t)} \phi(u_1, \ldots, u_{t-1}, b_j)\psi(v_1, \ldots, v_{t-1}, b_j)$$

Extend the map $\phi * \psi$ linearly to all of the domain to produce a $(t-1)$-linear map.

It is straigtforward to see that the above definition is well defined: the map is the same for any choice of orthonormal basis by the linearity of ϕ and ψ. A proof of this fact appears in [32].

Definition. *Let $V_1, \ldots, V_t$ be finite dimensional vector spaces over $\mathbb{R}$ and let $\phi : V_1 \times \cdots \times V_t \to \mathbb{R}$ be a t-linear map and let s be an integer $0 \leq s \leq t - 1$. Define*

$$\phi^{2^s} : V_1^{\otimes 2^s} \times \cdots \times V_{t-s}^{\otimes 2^s} \to \mathbb{R}$$

where $\phi^{2^0} := \phi$ and $\phi^{2^s} := \phi^{2^{s-1}} * \phi^{2^{s-1}}$.

Note that we only define this for exponents which are powers of two because the product $*$ is only defined when the domains of the maps are the same. An expression like $\phi^3 = \phi * (\phi * \phi)$ does not make sense because ϕ and $\phi * \phi$ have different domains. This defines the power $\phi^{2^{t-1}}$, which is a linear map $V_1^{\otimes 2^{t-1}} \to \mathbb{R}$.

Definition. *Let $V_1, \ldots, V_t$ be finite dimensional vector spaces over $\mathbb{R}$ and let $\phi : V_1 \times \cdots \times V_t \to \mathbb{R}$ be a t-linear map and define $A[\phi^{2^{t-1}}]$ to be the following square matrix/bilinear map. Let $u_1, \ldots, u_{2^{t-2}}, v_1, \ldots, v_{2^{t-2}}$ be vectors where $u_i, v_i \in V_1$.*

$$A[\phi^{2^{t-1}}] : V_1^{\otimes 2^{t-2}} \times V_1^{\otimes 2^{t-2}} \to \mathbb{R}$$

$$A[\phi^{2^{t-1}}](u_1 \otimes \cdots \otimes u_{2^{t-2}}, v_1 \otimes \ldots v_{2^{t-2}}) := \phi^{2^{t-1}}(u_1 \otimes v_1 \otimes u_2 \otimes v_2 \otimes \cdots \otimes u_{2^{t-2}} \otimes v_{2^{t-2}}).$$

Extend the map linearly to the entire domain to produce a bilinear map.

It is straigtforward to check that by definition, $A[\phi^{2^{t-1}}]$ is a square symmetric real-valued matrix for any ϕ; a proof of this fact appears in [32].

6.2 Counting walks and circuits

This section contains the proof of the following proposition.

Proposition 6. *Let H be a k-uniform hypergraph with loops, let $\vec{\pi}$ be a proper ordered partition of k, and let $\ell \geq 2$ be an integer. Let τ be the adjacency map of H. Then $Tr\left[A[\tau_{\vec{\pi}}^{2^{t-1}}]^\ell\right]$ is the number of labeled circuits of type $\vec{\pi}$ and length 2ℓ in H.*

The proof of this proposition comes down to showing that the function $\tau_{\vec{\pi}}^{2^{t-1}}$ counts the step $S_{\vec{\pi}}$. We do this by induction by describing exactly the hypergraph counted by $\tau_{\vec{\pi}}^{2^s}$, which is the following hypergraph.

Definition. *For $\vec{\pi} = (1, \ldots, 1)$ with t parts, let $0 \leq s \leq t-1$ and define the hypergraph $D_{\vec{\pi},s}$ as follows. Let $A_1, \ldots, A_{t-s}$ be disjoint sets of size 2^s where elements are labeled by binary strings of length s and let $B_{t-s+1}, \ldots, B_t$ be disjoint sets of size 2^{s-1} where elements are labeled by binary strings of length $s - 1$. The vertex set of $D_{\vec{\pi},s}$ is $A_1 \dot\cup \cdots \dot\cup A_{t-s} \dot\cup B_{t-s+1} \dot\cup \cdots \dot\cup B_t$. Make $a_1, \ldots, a_{t-s}, b_{t-s+1}, \ldots, b_t$ an edge of $D_{\vec{\pi},s}$ if $a_i \in A_i$, $b_j \in B_j$, the codes for $a_1, \ldots, a_{t-s}$*

are all equal, and the code for b_{t-s+j} is equal to the code formed by removing the jth bit of the code for a_1.

For a general $\vec{\pi} = (k_1, \dots, k_t)$, start with $D_{(1,\dots,1),s}$ and expand each vertex into the appropriate size; that is, a vertex in A_i is expanded into k_i vertices and each vertex in B_j is expanded into k_j vertices. In $D_{\vec{\pi},s}$, each vertex in A_i is labeled by a pair (c, z) where c is a bit string of length s and $z \in [k_i]$. We call z the expansion index of the vertex.

The hypergraph $D_{\vec{\pi},0}$ is a single edge and the hypergraph $D_{\vec{\pi},t-1}$ is by definition the step $S_{\vec{\pi}}$. The following lemma precisely formulates what we mean when we say that $\tau_{\vec{\pi}}^{2^s}$ counts the hypergraph $D_{\vec{\pi},s}$.

Lemma 7. *Let H be a k-uniform hypergraph with loops, $\vec{\pi}$ a proper ordered partition of k with $\vec{\pi} = (k_1, \dots, k_t)$, and let $0 \le s \le t-1$. Let W be the vector space over $\mathbb{R}$ of dimension $|V(H)|$ and let τ be the adjacency map of H. Let $A_1, \dots, A_{t-s}, B_{t-s+1}, \dots, B_t$ be the vertex sets in the definition of $D_{\vec{\pi},s}$ and let Δ be any map $A_1 \cup \dots \cup A_{t-s} \to V(H)$. Then $\tau_{\vec{\pi}}^s$ counts the number of labeled, possibly degenerate copies of $D_{\vec{\pi},s}$ extending Δ as follows.*
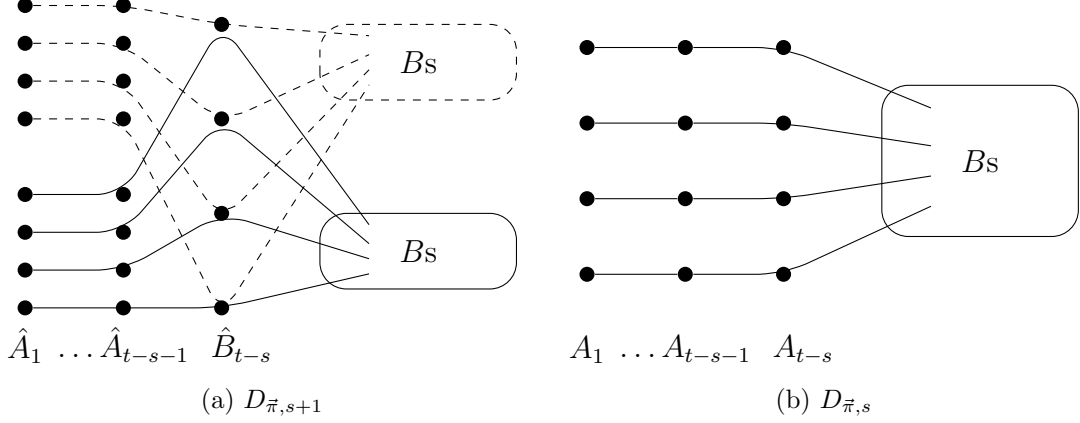
Let $a_{i,1}, \dots, a_{i,k_i 2^s}$ be the vertices of A_i ordered first lexicographically by bit code and then for equal codes ordered by expansion index. Let χ_i be the indicator tensor in $W^{\otimes k_i 2^s}$ for the vertex tuple $(\Delta(a_{i,1}), \dots, \Delta(a_{i,k_i 2^s}))$. Then $\tau_{\vec{\pi}}^{2^s}(\chi_1, \dots, \chi_{t-s})$ is the number of edge-preserving maps $V(D_{\vec{\pi},s}) \to V(H)$ which are consistent with Δ.

Proof. By induction on s. The base case is $s = 0$, where $D_{\vec{\pi},0}$ is a single edge, there are no B-type sets, and thus Δ is a map $V(D_{\vec{\pi},0}) \to V(H)$. The number of edge preserving maps extending Δ is either zero or one depending on if the image of Δ is an edge of H or not. But $\tau_{\vec{\pi}}(\chi_1, \dots, \chi_t)$ equals zero or one depending on if the vertices defining the indicator tensors χ_i form an edge, exactly what is required.

Assume the lemma is true for s; we will prove it for $s+1$. Denote by $\hat{A}_1, \dots, \hat{A}_{t-s-1}, \hat{B}_{t-s}, \dots, \hat{B}_t$ the sets in the definition of $D_{\vec{\pi},s+1}$ and $A_1, \dots, A_{t-s}, B_{t-s+1}, \dots, B_t$ the sets in the definition of $D_{\vec{\pi},s}$. Let $\hat{\Delta}$ be a map $\hat{A}_1 \cup \dots \cup \hat{A}_{t-s-1} \to V(H)$ and let $\hat{\chi}_1, \dots, \hat{\chi}_{t-s-1}$ be the indicator tensors for the image of $\hat{\Delta}$ ordered as in the statement of the lemma. Since $\hat{\chi}_i$ is an indicator tensor in $W^{\otimes k_i 2^{s+1}}$, it is a simple tensor so $\hat{\chi}_i = \chi_i \otimes \chi_i'$ for $\chi_i, \chi_i' \in W^{\otimes k_i 2^s}$. Note that χ_i is the indicator tensor for the image under $\hat{\Delta}$ of the vertices of $D_{\vec{\pi},s+1}$ whose code starts with zero and χ_i' is the indicator tensor for the image under $\hat{\Delta}$ of the vertices whose code starts with a one, since the definition of $\hat{\chi}_i$ sorted the vertices in the image lexicographically.

Consider the expansion of the definition of $\tau_{\vec{\pi}}^{2^{s+1}}(\hat{\chi}_1, \dots, \hat{\chi}_{t-s-1})$ shown in (3) in Figure 5; the tensors $\hat{\chi}_i$ are split into χ_i and χ_i' and we sum over the standard basis $\{w_1, \dots, w_d\}$ of $W^{\otimes k_{t-s} 2^s}$, where $d = \dim(W^{\otimes k_{t-s} 2^s})$. We can consider the tensor w_j in (3) to be the indicator tensor of a tuple of $k_{t-s} 2^s$ vertices.

Definition. *We now describe two embeddings of $D_{\vec{\pi},s}$ into $D_{\vec{\pi},s+1}$. In Figure 5 (a), these two embeddings are the dotted and solid lines. Let $\Gamma_0 : V(D_{\vec{\pi},s}) \to V(D_{\vec{\pi},s+1})$ be the following injection. For $1 \le i \le t-s-1$ and $a \in A_i$, set $\Gamma_0(a)$ equal to the vertex in $\hat{A}_i$ whose code equals the code for a with a zero prepended to the code and the same expansion index. That*

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(a) $D_{\vec{\pi}, s+1}$ (b) $D_{\vec{\pi}, s}$

$$\tau_{\vec{\pi}}^{2^{s+1}}(\chi_1 \otimes \chi_1', \dots, \chi_{t-s-1} \otimes \chi_{t-s-1}') = \sum_{j=1}^{d} \tau_{\vec{\pi}}^{2^s}(\chi_1, \dots, \chi_{t-s-1}, w_j) \tau_{\vec{\pi}}^{2^s}(\chi_1', \dots, \chi_{t-s-1}', w_j) \quad (3)$$

Figure 5: The induction step of Lemma 7

is, a vertex in A_i with label $(1011, 4)$ is mapped to the vertex in $\hat{A}_i$ with label $(01011, 4)$. For $a \in A_{t-s}$, set $\Gamma_0(a)$ equal to the vertex in $\hat{B}_{t-s}$ which has the same label as a. For $t - s + 1 \le j \le t$ and $b \in B_j$, set $\Gamma_0(b)$ equal to the vertex in $\hat{B}_j$ whose code equals the code for b with a zero prepended to the code and the same expansion index. In other words, Γ_0 adds a zero to the front of the codes except for vertices in A_{t-s} whose code does not change. Define $\Gamma_1 : V(D_{\vec{\pi}, s}) \to V(D_{\vec{\pi}, s+1})$ similarly except prepend a one instead of a zero. In Figure 5 (a), the dotted lines represent Γ_0 and the solid lines represent Γ_1.

Claim: Γ_0 and Γ_1 are edge preserving injections and every edge in $D_{\vec{\pi}, s+1}$ is in the image of Γ_0 or Γ_1 but not both.

Proof of Claim. Let E be an edge in $D_{\vec{\pi}, s}$. For $1 \le i \le j \le t - s - 1$ and $a_i \in A_i \cap E$ and $a_j \in A_j \cap E$, since E is an edge of $D_{\vec{\pi}, s}$ the code for a_i equals the code for a_j. This implies that the codes for $\Gamma_0(a_i)$ and $\Gamma_0(a_j)$ are equal since both had a zero prepended. Now consider $b \in A_{t-s} \cap E$ which is mapped to $\hat{B}_{t-s}$. The conditions for $\Gamma_0(E)$ an edge of $D_{\vec{\pi}, s+1}$ requires that the code for $\Gamma_0(b)$ equals the code formed by deleting the first bit of $\Gamma_0(a)$ where $a \in A_1 \cap E$. But the code for a equals the code for b since both are in A-type sets in $D_{\vec{\pi}, s}$ and the map Γ_0 adds a zero to the front of the code for a and leaves the code for b alone. Thus the code for $\Gamma_0(b)$ equals the code formed by deleting the first bit of $\Gamma_0(a)$. Lastly, consider $b \in B_j \cap E$ for $t - s + 1 \le j \le t$ and consider deleting the $(j+1)$-th bit of the code for $\Gamma_0(a)$. This is the same as deleting the j-th bit of a since $\Gamma_0(a)$ had a zero prepended. But deleting the jth bit of a equals the code for b, since $a, b \in E \in E(D_{\vec{\pi}, s})$. Thus deleting the $(j+1)$th bit of $\Gamma_0(a)$ is the code for $\Gamma_0(b)$. We have now checked all the conditions, so $\Gamma_0(E)$ is an edge of $D_{\vec{\pi}, s+1}$, i.e. Γ_0 is edge preserving. Γ_1 is edge preserving by

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the same argument. Finally, let E be an edge of $D_{\vec{\pi},s+1}$ and pick $a \in E \cap \hat{A}_1$. If the first bit of the code for a equals zero, then E is in the image of Γ_0 and if the first bit of the code for a equals one, then E is in the image of Γ_1. This concludes the proof of the claim. $\qquad \square$

This claim implies that any edge-preserving map extending $\hat{\Delta}$ is formed from two edge preserving maps $V(D_{\vec{\pi},s}) \to V(H)$ each extending the appropriate restriction of $\hat{\Delta}$. Start with $\hat{\Delta}$ and extend arbitrarily to a map $\Lambda : \hat{A}_1 \cup \cdots \cup \hat{A}_{t-s-1} \cup \hat{B}_{t-s} \to V(H)$. Next define Λ_0 and Λ_1 as maps $A_1 \cup \cdots \cup A_{t-s} \to V(H)$ such that $\Lambda_0 = \Lambda \circ \Gamma_0|_{\bar{A}}$ and $\Lambda_1 = \Lambda \circ \Gamma_1|_{\bar{A}}$, where $\bar{A} = A_1 \cup \cdots \cup A_{t-s}$ so $\Gamma_0|_{\bar{A}}$ is the map Γ_0 restricted to the A-type sets in $D_{\vec{\pi},s}$. By the claim, the number of edge-preserving maps extending $\hat{\Delta}$ equals the sum over Λ of the product of the number of edge-preserving maps extending Λ_0 and extending Λ_1. This is because any edge preserving map extending Λ can be composed with Γ_0 and Γ_1 to create edge preserving maps extending Λ_0 and Λ_1, and since Γ_0 and Γ_1 are injections covering all edges of $D_{\vec{\pi},s+1}$, this can be reversed. The last step in the proof is to show that this is exactly what (3) counts.

Let $\hat{b}_1, \ldots, \hat{b}_{k_{t-s}2^s}$ be the vertices of $\hat{B}_{t-s}$ listed first in lexicographic order of codes and then by expansion index. Let w be the indicator tensor in $W^{\otimes k_{t-s}2^s}$ for the vertex tuple $(\Lambda(\hat{b}_1), \ldots, \Lambda(\hat{b}_{k_{t-s}2^s}))$. Note that as Λ ranges over all possible extensions of $\hat{\Delta}$, w ranges over the standard basis of $W^{\otimes k_{t-s}2^s}$. Now $\chi_1, \ldots, \chi_{t-s-1}, w$ are the indicator tensors representing the image of the map Λ_0, since as mentioned above, $\chi_1, \ldots, \chi_{t-s-1}$ are the indicator tensors for the image under $\hat{\Delta}$ of the vertices whose code stars with a zero. Similarly, $\chi'_1, \ldots, \chi'_{t-s-1}, w$ are the indicator tensors representing the image of the map Λ_1. Thus by induction, $\tau_{\vec{\pi}}^{2^s}(\chi_1, \ldots, \chi_{t-s-1}, w)$ is the number of edge-preserving maps extending Λ_0 and $\tau_{\vec{\pi}}^{2^s}(\chi'_1, \ldots, \chi'_{t-s-1}, w)$ is the number of edge preserving maps extending Λ_1. By the claim, this implies that the product

$$\tau_{\vec{\pi}}^{2^s}(\chi_1, \ldots, \chi_{t-s-1}, w)\tau_{\vec{\pi}}^{2^s}(\chi'_1, \ldots, \chi'_{t-s-1}, w)$$

counts the number of edge-preserving maps extending Λ. Thus (3) sums over the choices for Λ extending $\hat{\Delta}$ of the number of edge-preserving maps extending Λ. This sum is exactly the number of edge-preserving maps extending $\hat{\Delta}$, so the proof is complete. $\qquad \square$

Corollary 8. *Let H be a k-uniform hypergraph with loops, $\vec{\pi}$ a proper ordered partition of k with $\vec{\pi} = (k_1, \ldots, k_t)$, and let $\ell \geq 2$ be an integer. Let W be the vector space over $\mathbb{R}$ of dimension $|V(H)|$ and let τ be the adjacency map of H. Let $a_1, \ldots, a_{k_1 2^{t-2}}, a'_1, \ldots, a'_{k_1 2^{t-2}}$ be (not necessarily distinct) vertices of H and let ξ and ξ' be the indicator tensors in $W^{k_1 2^{t-2}}$ for the tuples $(a_1, \ldots, a_{k_1 2^{t-2}})$ and $(a'_1, \ldots, a'_{k_1 2^{t-2}})$ respectively. Then $A[\tau_{\vec{\pi}}^{2^{t-1}}](\xi, \xi')$ is the number of labeled, possibly degenerate steps of type $\vec{\pi}$ in H with attach tuples $(a_1, \ldots, a_{k_1 2^{t-2}})$ and $(a'_1, \ldots, a'_{k_1 2^{t-2}})$. Also, $A[\tau_{\vec{\pi}}^{2^{t-1}}]^\ell(\xi, \xi')$ is the number of labeled walks of length 2ℓ and type $\vec{\pi}$ with attach tuples $(a_1, \ldots, a_{k_1 2^{t-2}})$ and $(a'_1, \ldots, a'_{k_1 2^{t-2}})$.*

Proof. The proof is by induction on ℓ. First, consider the base case of $\ell = 1$, where the path of length two and type $\vec{\pi}$ is the step of type $\vec{\pi}$. Let A be the vertex set from the definition of the step $S_{\vec{\pi}}$. Define a mapping $\Delta : A \to V(H)$ by mapping the attach tuples of $S_{\vec{\pi}}$ to the tuples

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$(a_1, \ldots, a_{k_1 2^{t-2}})$ and $(a'_1, \ldots, a'_{k_1 2^{t-2}})$ in $V(H)$. By definition, the first attach tuple of $S_{\vec{\pi}}$ is the vertices ending with a zero and listed in lexicographic order and the second attach tuple of $S_{\vec{\pi}}$ is the vertices ending with a one and listed in lexicographic order. This implies that the indicator tensor χ_1 from the statement of Lemma 7 is the indicator tensor in $W^{\otimes k_1 2^{t-1}}$ for the tuple $(a_1, \ldots, a_{k_1}, a'_1, \ldots, a'_{k_1}, a_{k_1+1}, \ldots, a_{2k_1}, a'_{k_1+1}, \ldots, a'_{2k_1}, \ldots, a_{k_1 2^{t-3}+1}, \ldots, a_{k_1 2^{t-2}}, a'_{k_1 2^{t-3}+1}, \ldots, a'_{k_1 2^{t-2}})$, since each attach tuple is in lexicographic order but the last bit is zero or one so the full ordering alternates between attach tuples. By the definition of $A[\tau_{\vec{\pi}}^{2^{t-1}}]$ and the indicator tensors ξ, ξ', χ_1, $A[\tau_{\vec{\pi}}^{2^{t-1}}](\xi, \xi') = \tau_{\vec{\pi}}^{2^{t-1}}(\chi_1)$. Thus Lemma 7 applied with $s = t - 1$ shows that the number of edge-preserving maps extending Δ is $A[\tau_{\vec{\pi}}^{2^{t-1}}](\xi, \xi')$, but by the definition of Δ, this is exactly the number of labeled, possibly degenerate steps of type $\vec{\pi}$ with attach tuples $(a_1, \ldots, a_{k_1 2^{t-2}})$ and $(a'_1, \ldots, a'_{k_1 2^{t-2}})$.

Next assume that the corollary is true for ℓ; we will show that it is true for $\ell + 1$. Using the definition of matrix multiplication, let $\{d_1, \ldots, d_{\dim(W^{\otimes k_1 2^{t-2}})}\}$ be the standard basis of $W^{\otimes k_1 2^{t-2}}$ so

$$A[\tau_{\vec{\pi}}^{2^{t-1}}]^{\ell+1}(\xi, \xi') = \sum_{i=1}^{\dim(W^{\otimes k_1 2^{t-2}})} A[\tau_{\vec{\pi}}^{2^{t-1}}]^{\ell}(\xi, d_i) A[\tau_{\vec{\pi}}^{2^{t-1}}](d_i, \xi'). \tag{4}$$

Each standard basis vector d_i can be thought of as a $k_1 2^{t-2}$-tuple of vertices which corresponds to one of the two attach tuples. Thus (4) sums over the internal attach tuple for a walk of length 2ℓ and $S_{\vec{\pi}}$. $\qquad\square$

Proof of Proposition 6. Since $A[\tau_{\vec{\pi}}^{2^{t-1}}]^{\ell}$ counts the number of walks of length 2ℓ, the trace counts circuits. If $\{d_1, \ldots, d_{\dim(W^{\otimes k_1 2^{t-2}})}\}$ is any orthonormal basis of $W^{\otimes k_1 2^{t-2}}$, the trace of the matrix $A[\tau_{\vec{\pi}}^{2^{t-1}}]^{\ell}$ is

$$\mathrm{Tr}\left[A[\tau_{\vec{\pi}}^{2^{t-1}}]^{\ell}\right] = \sum_{i=1}^{\dim(W^{\otimes k_1 2^{t-2}})} A[\tau_{\vec{\pi}}^{2^{t-1}}]^{\ell}(d_i, d_i).$$

If $\{d_1, \ldots, d_{\dim(W^{\otimes k_1 2^{t-2}})}\}$ is the standard basis, each d_i corresponds to a tuple of $k_1 2^{t-2}$ vertices, so the above expression is the number of walks of type $\vec{\pi}$ with both attach tuples equal to d_i. $\qquad\square$

6.3 Bounding eigenvalues from cycle counts

This section contains the proof that $\mathrm{Cycle}_{4\ell}[\pi] \Rightarrow \mathrm{Eig}[\pi]$ for d-coregular hypergraphs with loops. First, we require a few simple algebraic facts of multilinear maps.

Lemma 9. *Let $t \geq 2$, let $V_1, \ldots, V_t$ be finite dimensional vector spaces over $\mathbb{R}$, let $\phi : V_1 \times \cdots \times V_t \to \mathbb{R}$ be a t-linear map and let $x_1 \in V_1, \ldots, x_t \in V_t$ be unit length vectors. Then*

$$|\phi(x_1, \ldots, x_t)|^2 \leq \left|\phi^2(x_1 \otimes x_1, \ldots, x_{t-1} \otimes x_{t-1})\right|.$$

Proof. Consider the linear map $\phi(x_1, \ldots, x_{t-1}, \cdot)$ which is a linear map from V_t to $\mathbb{R}$. There exists a vector $w \in V_t$ such that $\phi(x_1, \ldots, x_{t-1}, \cdot) = \langle w, \cdot \rangle$. Then

$$\phi^2(x_1 \otimes x_1, \ldots, x_{t-1} \otimes x_{t-1}) = \sum_j |\phi(x_1, \ldots, x_{t-1}, b_j)|^2 = \sum_j |\langle w, b_j \rangle|^2 = \langle w, w \rangle$$

where the last equality is because $\{b_j\}$ is an orthonormal basis of V_t. Since $\|w\| = \sqrt{\langle w, w \rangle}$, $|\phi^2(x_1 \otimes x_1, \ldots, x_{t-1} \otimes x_{t-1})| = |\langle w, w \rangle| = |\langle w, w/\|w\| \rangle|^2$. But since x_t is unit length and $\langle w, \cdot \rangle$ is maximized over the unit ball at vectors parallel to w (so maximized at $\pm w/\|w\|$), $|\langle w, w/\|w\| \rangle| \geq |\langle w, x_t \rangle|$. Thus

$$\left| \phi^2(x_1 \otimes x_1, \ldots, x_{t-1} \otimes x_{t-1}) \right| = \left| \left\langle w, \frac{w}{\|w\|} \right\rangle \right|^2 \geq |\langle w, x_t \rangle|^2 = |\phi(x_1, \ldots, x_t)|^2 .$$

The last equality used the definition of w, that $\phi(x_1, \ldots, x_{t-1}, \cdot) = \langle w, \cdot \rangle$. $\square$

Lemma 10. *Let $t \geq 2$, let $V_1, \ldots, V_t$ be finite dimensional vector spaces over $\mathbb{R}$, and let $\phi : V_1 \times \cdots \times V_t \to \mathbb{R}$ be a t-linear map. Then for any unit length $x_1 \in V_1, \ldots, x_t \in V_t$, we have*

$$|\phi(x_1, \ldots, x_t)|^{2^{t-1}} \leq \left| A[\phi^{2^{t-1}}](\underbrace{x_1 \otimes \cdots \otimes x_1}_{2^{t-2}}, \underbrace{x_1 \otimes \cdots \otimes x_1}_{2^{t-2}}) \right| . \tag{5}$$

Also,

$$\|\phi\|^{2^{t-1}} \leq \lambda_1(A[\phi^{2^{t-1}}]). \tag{6}$$

Proof. By induction on s we have that

$$|\phi(x_1, \ldots, x_t)|^{2^s} \leq \left| \phi^{2^s}(\underbrace{x_1 \otimes \cdots \otimes x_1}_{2^s}, \ldots, \underbrace{x_{t-s} \otimes \cdots \otimes x_{t-s}}_{2^s}) \right|$$

Indeed, the base case is $s = 0$ where both sides are equal and the induction step follows from the previous lemma, since

$$\left(|\phi(x_1, \ldots, x_t)|^{2^{s-1}} \right)^2 \leq \left| \phi^{2^{s-1}}(\underbrace{x_1 \otimes \cdots \otimes x_1}_{2^{s-1}}, \ldots, \underbrace{x_{t-s+1} \otimes \cdots \otimes x_{t-s+1}}_{2^{s-1}}) \right|^2$$

$$\leq \left| \phi^{2^s}(\underbrace{x_1 \otimes \cdots \otimes x_1}_{2^s}, \ldots, \underbrace{x_{t-s} \otimes \cdots \otimes x_{t-s}}_{2^s}) \right| .$$

By definition of $A[\phi^{2^{t-1}}]$, $|A[\phi^{2^{t-1}}](x_1 \otimes \cdots \otimes x_1, x_1 \otimes \cdots \otimes x_1)| = |\phi^{2^{t-1}}(x_1 \otimes \cdots \otimes x_1)|$, completing the proof of (5). Let $x_1, \ldots, x_t$ as unit length vectors maximizing ϕ. Since $x_1 \otimes \cdots \otimes x_1$ is unit length, (5) proves that

$$\|\phi\|^{2^{t-1}} = |\phi(x_1, \ldots, x_t)|^{2^{t-1}} \leq \left| A[\phi^{2^{t-1}}](x_1 \otimes \cdots \otimes x_1, x_1 \otimes \cdots \otimes x_1) \right| \leq \lambda_1(A[\phi^{2^{t-1}}]).$$

$\square$

Corollary 11. *Let H be a d-coregular, k-uniform hypergraph with loops and let π be any proper partition of k with t parts. Then for any ordering $\vec{\pi}$ of π,*

$$\lambda_{2,\pi}(H) \le \left(\lambda_2(A[\tau_{\vec{\pi}}^{2^{t-1}}])\right)^{2^{-t+1}}.$$

Proof. Let $\vec{\pi} = (k_1, \ldots, k_t)$ and let $x_1, \ldots, x_t$ be unit length vectors maximizing $\tau_{\vec{\pi}} - \frac{d}{n}J_{\vec{\pi}}$ in absolute value, so that $\lambda_{2,\pi}(H) = \left|(\tau_{\vec{\pi}} - \frac{d}{n}J_{\vec{\pi}})(x_1, \ldots, x_t)\right|$. Write $x_1 = \alpha y + \beta \hat{1}$, where y is a unit length vector perpendicular to the all-ones vector, $\hat{1}$ is the all-ones vector scaled to unit length, and $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 = 1$. Let W be the vector space over $\mathbb{R}$ of dimension n and for $1 \le i \le t$ let $e_{i,1}, \ldots, e_{i,n^{k_i}}$ be the standard basis of $W^{\otimes k_i}$. Since H is d-coregular,

$$\tau_{\vec{\pi}}(\hat{1}, x_2, \ldots, x_t) = \frac{1}{n^{k/2}} \sum_{1 \le j_2 \le n^{k_2}} \cdots \sum_{1 \le j_t \le n^{k_t}} \langle e_{2,j_2}, x_2 \rangle \cdots \langle e_{t,j_t}, x_t \rangle \sum_{1 \le j_1 \le n^{k_1}} \tau_{\vec{\pi}}(e_{1,j_1}, \ldots, e_{t,j_t})$$

$$= \frac{1}{n^{k/2}} \sum_{1 \le j_2 \le n^{k_2}} \cdots \sum_{1 \le j_t \le n^{k_t}} \langle e_{2,j_2}, x_2 \rangle \cdots \langle e_{t,j_t}, x_t \rangle \, dn^{k_1-1} J_{\vec{\pi}}(e_{1,1}, e_{2,j_2}, \ldots, e_{t,j_t})$$

$$= \frac{d}{n} J_{\vec{\pi}}(\hat{1}, x_2, \ldots, x_t).$$

Next, $J_{\vec{\pi}}(y, x_2, \ldots, x_t) = \langle 1, y \rangle \langle 1, x_2 \rangle \cdots \langle 1, x_t \rangle$. Since y is perpendicular to the all-ones vector, $J_{\vec{\pi}}(y, x_2, \ldots, x_t) = 0$. Therefore, using linearity,

$$\lambda_{2,\pi}(H) = \left|\left(\tau_{\vec{\pi}} - \frac{d}{n}J_{\vec{\pi}}\right)(\alpha y + \beta \hat{1}, x_2, \ldots, x_t)\right| = |\alpha|\,|\tau_{\vec{\pi}}(y, x_2, \ldots, x_t)|.$$

By (5) applied to $\tau_{\vec{\pi}}(y, x_2, \ldots, x_t)$, $\lambda_{2,\pi}(H) \le |\alpha|\,|A[\tau_{\vec{\pi}}^{2^{t-1}}](y \otimes \cdots \otimes y, y \otimes \cdots \otimes y)|^{2^{-t+1}}$. Since H is d-coregular, the number of steps of type $\vec{\pi}$ with a fixed attach tuple $A^{(0)}$ is independent of the choice of $A^{(0)}$. By Corollary 8, each row of the matrix $A[\tau_{\vec{\pi}}^{2^{t-1}}]$ cooresponds to an attach tuple $A^{(0)}$ and the sum of the entries in that row counts the number of steps of type $\vec{\pi}$ with fixed attach tuple $A^{(0)}$. Therefore, each row of $A[\tau_{\vec{\pi}}^{2^{t-1}}]$ sums to the same value so that the Perron-Frobenius Theorem implies that the all-ones vector is the eigenvector associated to $\lambda_1(A[\tau_{\vec{\pi}}^{2^{t-1}}])$. Since $y \otimes \cdots \otimes y$ is perpendicular to the all-ones vector and $A[\tau_{\vec{\pi}}^{2^{t-1}}]$ is a square real symmetric matrix, $|A[\tau_{\vec{\pi}}^{2^{t-1}}](y \otimes \cdots \otimes y, y \otimes \cdots \otimes y)| \le \lambda_2(A[\tau_{\vec{\pi}}^{2^{t-1}}])$. Since $|\alpha| \le 1$, the proof is complete. $\square$

Proof that `Cycle₄ₗ[π]` $\Rightarrow$ `Eig[π]`. Let $\mathcal{H} = \{H_n\}_{n \to \infty}$ be a sequence of d-coregular, k-uniform hypergraphs with loops and let τ_n be the adjacency map of H_n. For notational convenience, the subscript on n is dropped below. Let $\vec{\pi}$ be any ordering of the entries of π. Let $m = |E(C_{\pi,4\ell})| = 2\ell 2^{t-1}$ and note that $|V(C_{\pi,4\ell})| = mk/2$ since $C_{\pi,4\ell}$ is two-regular. The matrix $A = A[\tau_{\vec{\pi}}^{2^{t-1}}]$ is a square symmetric real valued matrix, so let $\mu_1, \ldots, \mu_r$ be the eigenvalues of A arranged so that $|\mu_1| \ge \cdots \ge |\mu_r|$, where $r = \dim(A)$. The eigenvalues of $A^{2\ell}$ are $\mu_1^{2\ell}, \ldots, \mu_d^{2\ell}$ and the trace of $A^{2\ell}$ is $\sum_i \mu_i^{2\ell}$. Since all $\mu_i^{2\ell} \ge 0$, Proposition 6 and `Cycle₄ₗ[π]` imply that

$$\mu_1^{2\ell} \le \mu_1^{2\ell} + \mu_2^{2\ell} \le \text{Tr}\left[A^{2\ell}\right] = \#\{\text{possibly degen } C_{\pi,4\ell} \text{ in } H_n\} \le p^m n^{mk/2} + o(n^{mk/2}). \quad (7)$$

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Since $pn^{k/2} = dn^{k/2-1} = \tau_{\vec{\pi}}(\hat{1}, \ldots, \hat{1}) \leq \|\tau_{\vec{\pi}}\| = \lambda_{1,\pi}(H)$, (6) implies that $\mu_1 \geq p^{2^{t-1}} n^{k2^{t-2}}$ which implies equality up to $o(n^{mk/2})$ throughout (7). Therefore, $\mu_2 = o(n^{k2^{t-2}})$ so that Corollary 11 shows that $\lambda_{2,\pi}(H) = o(n^{k/2})$, completing the proof. $\qquad\square$

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