Ramsey numbers of cliques versus monotone paths

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Abstract

One formulation of the Erdős-Szekeres monotone subsequence theorem states that for any red/blue coloring of the edge set of the complete graph on $\{1, 2, \ldots, N\}$, there exists a monochromatic red s-clique or a monochromatic blue increasing path P_n with n vertices, provided $N > (s-1)(n-1)$. Here, we prove a similar statement as above in the off-diagonal case for triple systems, with the quasipolynomial bound $N > 2^{c(\log n)^{s-1}}$. For the tth power P_n^t of the ordered increasing graph path with n vertices, we prove a near linear bound $cn(\log n)^{s-2}$ which improves the previous bound that applied to a more general class of graphs than P_n^t due to Conlon-Fox-Lee-Sudakov.

1 Introduction

A well-known theorem of Erdős and Szekeres [11] states that any sequence of $(n-1)^2 + 1$ distinct real numbers contains a monotone subsequence of length at least n. This is a classical result in combinatorics and its generalizations and extensions have many important consequences in geometry, probability, and computer science. See Steele [22] for 7 different proofs along with several applications. Here, we study its extension in the ordered hypergraph setting.

An *ordered* k-uniform hypergraph H on n vertices is a hypergraph whose vertices are ordered $\{1, 2, \ldots, n\}$. Given two ordered k-uniform hypergraphs G and H, the Ramsey numbers $r_k(G, H)$ is the minimum N such that for every red/blue coloring of the k-tuples of $\{1, 2, \ldots, N\}$, there is either a red copy of G or a blue copy of H. When $G = H$, we simply write $r_k(H) = r_k(H, H)$. We let $r_k(H; q)$ to be the minimum integer N such that for every q-coloring of the k-tuples of $[N] = \{1, 2, \ldots, N\}$, there is a monochromatic copy of H. We write $K_n^{(k)}$ for the complete k-uniform hypergraph on n vertices. A *monotone path of size* n, denoted by $P_n^{(k)}$, is an ordered k-uniform hypergraph whose vertex set is $\{1, 2, \ldots, n\}$, and $n-k+1$ edges of the form $(i, i+1, \ldots, i+k-1)$, for $i = 1, \ldots, n-k+1$. In order to avoid the excessive use of superscripts, we remove them when the uniformity is clear. For example, we write $r_k(K_s, P_n) = r_k(K_s^{(k)}, P_n^{(k)})$.

The proof of the Erdős and Szekeres monotone subsequence theorem, and also Dilworth's theorem on partially ordered sets [6], implies that

$$r_2(K_s, P_n) = (s-1)(n-1) + 1.$$

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However for k-uniform hypgeraphs, when $k \geq 3$, $r_k(K_s, P_n)$ is much less understood. In [17], the authors showed a surprising connection between $r_k(K_s, P_n)$ and the classical Ramsey number $r_{k-1}(K_s; q)$. More precisely, they showed that for $q \geq 2$

$$r_{k-1}(K_{\lfloor s/q \rfloor}; q) \leq r_k(K_s, P_{q+k-1}) \leq r_{k-1}(K_s; q). \tag{1}$$

Hence, for $q = 2$, $k = O(1)$, and s tending to infinity, determining the tower growth rate of $r_k(K_s, P_{k+1})$ is equivalent to determining the tower growth rate of the classical Ramsey number $r_{k-1}(K_s)$. Classical results of Erdős [7] and Erdős and Szekeres [11] imply that $r_2(K_s) = 2^{\Theta(s)}$ (see also [21, 19, 3]). Unfortunately for k-uniform hypergraphs, when $k \geq 3$, there is an exponential gap between the best known lower and upper bounds for $r_k(K_s)$. More precisely,

$$\mathrm{twr}_{k-1}(\Omega(s^2)) < r_k(K_s) < \mathrm{twr}_k(O(s)),$$

where the tower function $\mathrm{twr}_k(x)$ is defined recursivly by $\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}$ (see [8, 9, 10]). A notoriously difficult conjecture of Erdős, Hajnal, and Rado states that the upper bound is the correct tower growth rate.

Unfortunately, (1) doesn't shed much light on $r_k(K_s, P_n)$ when s is fixed, and n tends to infinity. In this direction, the first author [16] showed that $r_3(K_4, P_n) = O(n^{21})$ and made the following conjecture.

Conjecture 1.1. *We have $r_3(K_s, P_n) = O(n^c)$, where $c = c(s)$.*

Our first result establishes a quasi-polynomial bound for $r_3(K_s, P_n)$, when s is fixed. Throughout this paper, all logarithms are in base 2.

Theorem 1.2. *We have $r_3(K_s, P_n) < 2^{c_s (\log n)^{s-1}}$, where $c_s = 5^s s!$.*

Together with the well-known neighborhood chasing argument of Erdős and Rado [10], we have the following.

Theorem 1.3. *For $k \geq 3$, we have $r_k(K_s, P_n) = \mathrm{twr}_{k-2}\left(2^{c(\log n)^{s-1}}\right)$, where $c = c(s)$.*

In the other direction, we have the trivial inequality $r_k(K_s, P_n) \geq r_k(P_s, P_n)$. The famous cups-caps theorem of Erdős and Szekeres [11] states that $r_3(P_s, P_n) = \binom{s+n-4}{s-2} + 1$, and the stepping-up lemma established in [12] (see Theorem 4.3) implies that $r_k(P_s, P_n) \geq \mathrm{twr}_{k-2}(n^c)$, where $c = c(s)$. Thus, we essentially determine the tower growth rate of $r_k(K_s, P_n)$ for s fixed and n tending to infinity.

For the diagonal case $r_3(K_n, P_n)$, these observations and a result of the authors [10] yield

$$2^n < \binom{2n-4}{n-2} = r_3(P_n, P_n) \leq r_3(K_n, P_n) < r_2(n; n) < 2^{n^2 \log n}.$$

It would be interesting to improve either bound for $r_3(K_n, P_n)$.

1.1 Cliques versus power paths in graphs

A key lemma in the first author's [16] proof of $r_3(K_4, P_n) = O(n^{21})$ is based on the following generalization of monotone paths in ordered graphs. Given positive integers t, n, the *t-th power of the path of P_n*, denoted by P_n^t, is an ordered graph with vertex set $\{1, 2, \ldots, n\}$, and (i, j) is an edge if and only if $|j-i| \leq t$. Hence, $P_n^1 = P_n$. In [2], Balko et al. showed that $r_2(P_n^t) = O(n^{129t})$ (see also [16]). Our next result establishes a near linear bound in the off-diagonal setting. Moreover, our proof generalizes to the clique versus power-path setting.

Theorem 1.4. *For positive integers s, t, n such that $t \leq s$, we have*

$$r_2(P_s^t, P_n^t) \leq r_2(K_s, P_n^t) < t^{4s} n (\log n)^{s-2}.$$

For large s, e.g., $s = n$, we also have the following bound.

Theorem 1.5. *For positive integers s, t, n, we have*

$$r_2(K_s, P_n^t) < (2s)^{t(t+1)\log n}.$$

Hence in the diagonal setting, for fixed $t > 0$, we have $r_2(K_n, P_n^t) \leq 2^{O(\log^2 n)}$. This coincides with a more general result established by Conlon, Fox, Lee, and Sudakov [4] on ordered graphs with bounded degeneracy. In the off-diagonal case, we make the following stronger conjecture.

Conjecture 1.6. *For all $s, t > 1$ there exists $c = c_{s,t}$ such that $r_2(K_s, P_n^t) < cn$.*

2 Non-increasing sets: Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by establishing a Ramsey-type result for non-increasing sets. Let χ be a q-coloring of the pairs of $[N]$, with colors $\{\kappa_1, \ldots, \kappa_q\} \subset \mathbb{Z}$ such that $\kappa_1 < \cdots < \kappa_q$. Then we say that a triple $u, v, w \in [N]$, where $u < v < w$, is *non-increasing* if

1. $\chi(u,v) = \chi(u,w) \geq \chi(v,w)$, or
2. $\chi(u,v) \geq \chi(u,w) = \chi(v,w)$.

We say that a set $S \subset [N]$ is *non-increasing* with respect to χ if every triple in S is non-increasing. Given subsets $S, T \subset [N]$ such that $S = \{v_1, \ldots, v_s\}$ and $T = \{u_1, \ldots, u_s\}$, we say that S and T have the same *color pattern* with respect to χ if $\chi(v_i, v_j) = \chi(u_i, u_j)$ for all i, j.

We will need the following lemma about non-increasing sets.

Lemma 2.1. *Let $S = \{v_1, \ldots, v_s\}$ be a non-increasing set with respect to χ, where $v_1 < \cdots < v_s$. Fix vertex $v_j \in S$. Then for any $v_i, v_\ell \in S$ such that $v_i < v_j < v_\ell$, we have*

1. *$\chi(v_i, v_j) \geq \chi(v_j, v_\ell)$, and*
2. *$\chi(v_{j-1}, v_j) \leq \chi(v_i, v_j)$, and*
3. *$\chi(v_j, v_{j+1}) \geq \chi(v_j, v_\ell)$.*

Proof. The first property follows from the fact that S is non-increasing. For the second property, for sake of contradiction, suppose there is a vertex $v_i < v_{j-1}$ such that $\chi(v_{j-1}, v_j) > \chi(v_i, v_j)$. Then we must have $\chi(v_i, v_{j-1}) = \chi(v_i, v_j)$, contradicting the fact that $\{v_i, v_{j-1}, v_j\}$ is non-increasing. A similar argument shows that the third property follows. □

Let $f(s; q)$ be the minimum integer N, such that if the pairs of $[N]$ are colored with at most q colors $\kappa_1 < \cdots < \kappa_q$, then there is a set $S \subset [N]$ of size s such that every triple in S is non-increasing.

Theorem 2.2. *We have* $r_3(K_s, P_n) \le f(s; n-2)$.

Proof. Let $N = f(s;n)$ and let ϕ be a red-blue coloring of the triples of $[N]$. If ϕ produces a blue monotone path of size n, then we are done. Otherwise, we define $\chi : \binom{[N]}{2} \to \{2, 3, \ldots, n-1\}$ such that for $u, v \in [N]$, $\chi(u,v)$ is the size of the longest blue monotone path ending at (u,v) with respect to ϕ. Note that if there are no blue edges ending at (u,v), then $\chi(u,v) = 2$. By definition of $f(s;n)$, there is a set $S \subset [N]$ of s vertices such that every triple in S is non-increasing with respect to χ. Notice that if a triple $u, v, w \in S$, where $u < v < w$, is colored blue with respect to ϕ, then the longest monotone path ending at (u,v) could be extended to a longer monotone path ending at (v,w), contradicting the fact that S is non-increasing. Hence, ϕ must color every triple in S red, which yields a red K_s with respect ϕ. □

We now prove the following upper bound for $f(s;n)$. Together with Theorem 2.2, Theorem 1.2 quickly follows.

Theorem 2.3. *For* $s \ge 3$ *and* $n \ge 2$*, we have* $f(s;n) \le 2^{5^s s! (\log n)^{s-2}}$.

Proof. We proceed by double induction on s and n. For the base case $n = 2$ and $s \ge 3$, we have

$$f(s;2) \le r_2(K_s) < 4^s < 2^{5^s s!}.$$

Therefore, let us assume that the statement holds for $n' < n$. For the other base case $s = 3$ and $n \ge 2$, let $N = 2^{5^3 \cdot 6 \log n}$ and χ be an n-coloring of the pairs (edges) of $[N]$ with colors $\{1, \ldots, n\}$. We can assume at least half of the edges have color $i \le n/2$, since otherwise a symmetric argument would follow. Let $E \subset \binom{[N]}{2}$ be the set of edges with color at most $n/2$, and for $v \in [N]$, let

$$N_E^-(v) = \{u \in [N] : u < v, (u,v) \in E\},$$

and $d_E^-(v) = |N_E^-(v)|$. Hence, $\sum_v d_E^-(v) = |E| \ge (1/2)\binom{N}{2}$.

By averaging, there is a vertex $v \in [N]$ such that $d_E^-(v) \ge (N-1)/4$. If there is a pair in $N_E^-(v)$ with color $j > n/2$, then we have a non-increasing triple and we are done. On the other hand, if no such pair has color $j > n/2$, since we have

$$|N_E^-(v)| \ge \frac{N-1}{4} > 2^{5^3 \cdot 6 \log(n/2)},$$

we can apply induction to find a non-increasing triple and we are done.

For the inductive step, let us assume that the statement holds for $s' < s$ and $n' < n$. Let $N = 2^{5^s s! (\log n)^{s-2}}$. Let χ be an n-coloring of the pairs of $[N]$ with colors $\{1, \ldots, n\}$. By a standard supersaturation argument, we have at least

$$\frac{\binom{N}{f(s-1;n)}}{\binom{N-(s-1)}{f(s-1;n)-(s-1)}} \ge \frac{(N-s)^{s-1}}{f(s-1;n)^{s-1}} \ge \frac{N^{s-1}}{2f(s-1;n)^{s-1}},$$

copies of a non-increasing set on $s-1$ vertices. By the pigeonhole principle, there are at least $N^{s-1}/(2n^{s^2} f(s-1;n)^{s-1})$ non-increasing sets on $s-1$ vertices with the same color pattern. Let us fix one such non-increasing set $S = \{v_1, \ldots, v_{s-1}\}$ for reference, and let $\chi(v_i, v_{i+1}) = \kappa_i$. For convenience, set $\kappa_0 = n$ and $\kappa_{s-1} = 1$, which implies

$$n = \kappa_0 \ge \kappa_1 \ge \cdots \ge \kappa_{s-2} \ge \kappa_{s-1} = 1.$$

By the pigeonhole principle, there is an i such that $1 \leq i \leq s-1$ such that $\kappa_{i-1} - \kappa_i \geq n/s$. Since we have $N^{s-1}/(2n^{s^2} f(s-1;n)^{s-1})$ non-increasing sets on $s-1$ vertices with the same color pattern as S, there is a subset $B \subset [N]$ and $s-2$ vertices $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{s-1} \in [N]$ such that for each $b \in B$, we have

1. $u_1 < \cdots < u_{i-1} < b < u_{i+1} < \cdots < u_{s-1}$,
2. $|B| \geq N/(2n^{s^2} f(s-1;n)^{s-1})$, and
3. $S' = \{u_1, \ldots, u_{i-1}, b, u_{i+1}, \ldots, u_{s-1}\}$ is non-increasing with the same color pattern as S.

Let us remark that if $i = 1$, then we have $b < u_2 < \cdots < u_{s-1}$ for all $b \in B$, and $S' = \{b, u_2, \ldots, u_{s-1}\}$. Likewise, if $i = s-1$, then we have $u_1 < \cdots < u_{s-2} < b$ for all $b \in B$, and $S' = \{u_1, \ldots, u_{s-2}, b\}$.

If there is a pair $b, b' \in B$ such that $\kappa_{i-1} \geq \chi(b, b') \geq \kappa_i$, then the set

$$T = \{u_1, \ldots, u_{i-1}, b, b', u_{i+1}, \ldots, u_{s-1}\}$$

is a nonincreasing set of size s. Indeed, it suffices to check that triples of the form $\{u_j, b, b'\}$ for $j \leq i-1$, and $\{b, b', u_j\}$ where $j \geq i+1$, are non-increasing. Assume $j \leq i-1$. By construction, we have $\chi(u_j, b) = \chi(u_j, b')$. By Lemma 2.1 and the assumption above, we have

$$\chi(u_j, b) = \chi(u_j, b') \geq \kappa_{i-1} \geq \chi(b, b').$$

Hence, $\{u_j, b, b'\}$ is non-increasing. For $j \geq i+1$, a similar argument shows that $\{b, b', u_j\}$ is non-increasing.

Therefore, we can assume that χ uses at most $n - n/s = n(s-1)/s$ distinct colors on B. However, this implies

$$\begin{aligned}
|B| &\geq \frac{N}{2n^{s^2} f(s-1;n)^{s-1}} \\
&\geq \frac{2^{5^s s!(\log n)^{s-2}}}{2n^{s^2} 2^{(s-1)5^{s-1}(s-1)!(\log n)^{s-3}}} \\
&\geq 2^{5^s s!(\log n)^{s-2} - 2(s-1)5^{s-1}(s-1)!(\log n)^{s-3}} \\
&\geq 2^{5^s s!(\log n - \log(s/(s-1)))^{s-2}} \\
&\geq 2^{5^s s!(\log((s-1)n/s))^{s-2}} \\
&\geq f(s; (s-1)n/s).
\end{aligned}$$

By the induction hypothesis, we can find a non-increasing set inside of B.

□

3 Ordered graphs

Proof of Theorem 1.4. We proceed by double induction on s and n. The base cases when $s = 2$ or when $n = 2$ is trivial. For the inductive step, assume that the statement holds for $s' < s$ or $n' < n$. Let $N = t^{4s}n(\log n)^{s-2}$, and $V = [N]$. For sake of contradiction, suppose there is $\chi : \binom{[N]}{2} \to \{\text{red,blue}\}$, such that χ does not produce a red K_s nor a blue P_n^t. Then we define

- $U = \{\lfloor N/2 \rfloor + 1, \lfloor N/2 \rfloor + 2, \ldots, \lfloor N/2 \rfloor + \binom{s+t}{t}\}$,
- $V_1 = \{1, 2, \ldots, \lfloor N/2 \rfloor\}$,
- $V_2 = \{\lfloor N/2 \rfloor + \binom{s+t}{t} + 1, \lfloor N/2 \rfloor + \binom{s+t}{t} + 2, \ldots, N\}$

By Ramsey's theorem, we know that $r_2(K_s, K_t) < \binom{s+t}{t}$. Hence, since $|U| = \binom{s+t}{t}$, we can conclude that U contains a blue K_t on vertices $u_1, \ldots, u_t \in U$. For $u_i \in U$, let

$$N_r(u_i) = \{v \in V : \chi(u_i, v) = \text{red}\}.$$

Then we have $|N_r(u_i)| < r_2(K_{s-1}, P_n^t)$. Let

$$V_1' = V_1 \setminus (N_r(u_1) \cup \cdots \cup N_r(u_t)),$$

$$V_2' = V_2 \setminus (N_r(u_1) \cup \cdots \cup N_r(u_t)).$$

Then notice that we must have either $|V_1'| < r_2(K_s, P^t_{\lfloor n/2 \rfloor})$ or $|V_2'| < r(K_s, P^t_{\lfloor n/2 \rfloor})$. Indeed, otherwise both V_1' and V_2' contain a blue $P^t_{\lfloor n/2 \rfloor}$. Since χ colors all edges between u_i and $V_1' \cup V_2'$ blue, we can combine both blue copies of $P^t_{\lfloor n/2 \rfloor}$ with vertices $u_1, \ldots, u_t$ and obtain a blue $P_{2\lfloor n/2 \rfloor + t}$, which contains a copy of a blue P_n^t since $2\lfloor n/2 \rfloor + t > n$.

Therefore, without loss of generality, we can assume that $|V_1'| < r_2(K_s, P^t_{\lfloor n/2 \rfloor})$. On the other hand, we have

$$|V_1'| \geq \lfloor N/2 \rfloor - \binom{s+t}{t} - t \cdot r_2(K_{s-1}, P_n^t).$$

Hence

$$N \leq 2r_2(K_s, P^t_{\lfloor n/2 \rfloor}) + 2\binom{s+t}{t} + 2t \cdot r_2(K_{s-1}, P_n^t).$$

By the induction hypothesis, we have

$$N \leq t^{4s}n(\log n - 1)^{s-2} + 2 \cdot 4^s + 2t \cdot t^{4s-4}n(\log n)^{s-3}.$$

$$\leq t^{4s}n(\log n)^{s-2} - (s-2)t^{4s}n(\log n)^{s-3} + (s-2)^2 t^{4s}n(\log n)^{s-4} + 2 \cdot 4^s + 2t^{4s-3}n(\log n)^{s-3}$$

$$\leq t^{4s}n(\log n)^{s-2}.$$

□

The proof of Theorem 1.5 is very similar to the argument above.

Proof of Theorem 1.5. We proceed by induction on n. The base case $n = 2$ is trivial. Now assume that the statement holds for all $n' < n$. Set $N = (2s)^{t(t+1)\log n}$. We start with a standard supersaturation argument. For sake of contradiction, suppose there is a red/blue coloring $\chi : \binom{[N]}{2} \rightarrow \{\text{red,blue}\}$ of the pairs of $[N]$ such that χ does not produce a red K_s nor a blue P_n^t. Let $r = r(K_s, K_{t+1})$. Then we must have at least

$$\frac{\binom{N}{r}}{\binom{N-(t+1)}{r-(t+1)}} = \frac{N!}{r!}\frac{(r-(t+1))!}{(N-(t+1))!} \geq \frac{(N-t)^{t+1}}{r^{t+1}} \geq \frac{N^{t+1}}{(2r)^{t+1}}$$

copies of K_{t+1}. For each blue copy of K_{t+1} with vertex set $x_0 < x_1 < \cdots < x_t$, we associate the middle $t-1$ vertices $\{x_1, \ldots, x_{t-1}\}$. By the pigeonhole principle, there is a set $Y = \{x_1, x_2, \ldots, x_{t-1}\}$ with $x_1 < x_2 < \cdots < x_{t-1}$, such that Y is the middle set for at least

$$\frac{N^{t+1}}{(2r)^{t+1}}\frac{1}{N^{t-1}} \geq \frac{N^2}{(2r)^{t+1}}$$

blue copies of K_{t+1}. Let $V_1 \subset \{1, 2, \ldots, x_1 - 1\}$ such that $x \in V_1$ if there is a blue K_{t+1} whose middle set is Y and x is the first vertex of the blue K_{t+1}. Likewise, let $V_2 \subset \{x_{t-1}+1, \ldots, N\}$ such that $x \in V_1$ if there is a blue K_{t+1} whose middle set is Y and x is the last vertex of the blue K_{t+1}. Hence, we have

$$|V_1||V_2| \geq \frac{N^2}{(2r)^{t+1}}.$$

Moreover, χ colors all edges between V_1 and Y blue, and all edges between V_2 and Y blue. Since $|V_1|, |V_2| < N$, we must have $|V_1|, |V_2| \geq \frac{N}{(2r)^{t+1}}$. Since the Erdős-Szekeres theorem implies that $r_2(K_s, K_{t+1}) \leq \binom{s+t-1}{t} \leq s^t$, we have

$$\min\{|V_1|, |V_2|\} \geq \frac{N}{(2s)^{t(t+1)}} = \frac{(2s)^{t(t+1)\log n}}{(2s)^{t(t+1)}} \geq (2s)^{t(t+1)\log\lfloor n/2\rfloor}.$$

By the inductive hypothesis, both V_1 and V_2 contain a blue $P^t_{\lfloor n/2\rfloor}$. Together with the vertices in Y, we obtain a blue copy of $P^t_{2\lfloor n/2\rfloor+t-1}$. Since $2\lfloor n/2\rfloor + t - 1 \geq n$, this completes the proof. □

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