A UNIFIED APPROACH TO HYPERGRAPH STABILITY

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ABSTRACT. We present a method which provides a unified framework for most stability theorems that have been proved in graph and hypergraph theory. Our main result reduces stability for a large class of hypergraph problems to the simpler question of checking that a hypergraph $\mathcal{H}$ with large minimum degree that omits the forbidden structures is vertex-extendable. This means that if v is a vertex of $\mathcal{H}$ and $\mathcal{H} - v$ is a subgraph of the extremal configuration(s), then $\mathcal{H}$ is also a subgraph of the extremal configuration(s). In many cases vertex-extendability is quite easy to verify.

It follows that our approach by giving new short proofs of hypergraph stability results of Pikhurko, Hefetz-Keevash, Brandt-Irwin-Jiang, Bene Watts-Norin-Yepremyan and others. Since our method always yields minimum degree stability, which is the strongest form of stability, in some of these cases our stability results are stronger than what was known earlier. Along the way, we clarify the different notions of stability that have been previously studied.

§1. INTRODUCTION

1.1. **Types of stability.** For $r \geqslant 2$ and a family $\mathcal{F}$ of r-uniform hypergraphs (henceforth called r-graphs), an r-graph $\mathcal{H}$ is said to be $\mathcal{F}$-*free* if it contains no member of $\mathcal{F}$ as a subgraph. For $n \in \mathbb{N}$ the *Turán number* $\mathrm{ex}(n, \mathcal{F})$ of $\mathcal{F}$ is the maximum number of edges in an $\mathcal{F}$-free r-graph on n vertices. The *Turán density* $\pi(\mathcal{F})$ of $\mathcal{F}$ is defined as $\pi(\mathcal{F}) = \lim_{n \to \infty} \mathrm{ex}(n, \mathcal{F}) / \binom{n}{r}$, and $\mathcal{F}$ is called *nondegenerate* if $\pi(\mathcal{F}) > 0$. The study of $\mathrm{ex}(n, \mathcal{F})$ is perhaps the central topic in extremal graph and hypergraph theory.

Much is known about $\mathrm{ex}(n, \mathcal{F})$ when $r = 2$ and one of the most famous results in this regard is Turán's theorem [37], which states that for $n \geqslant \ell \geqslant 2$ there is a unique $K_{\ell+1}$-free graph with n vertices and $\mathrm{ex}(n, K_{\ell+1})$ edges, namely the balanced complete ℓ-partite graph $T(n, \ell)$.

Turán's theorem was extended further by Erdős and Stone [10] in the following way. Given a family of graphs $\mathcal{F}$ the *chromatic number* $\chi(\mathcal{F})$ of $\mathcal{F}$ is defined as

$$\chi(\mathcal{F}) = \min\{\chi(F) \colon F \in \mathcal{F}\},$$

where $\chi(F)$ is the chromatic number of the graph F. The result of Erdős and Stone implies $\pi(\mathcal{F}) = 1 - \frac{1}{\chi(\mathcal{F})-1}$ for every family $\mathcal{F}$ of graphs; the connection to the chromatic number was first stated explicitly by Erdős and Simonovits [8].

Extending Turán's theorem to hypergraphs (i.e. $r \geqslant 3$) is a major problem. For $\ell > r \geqslant 3$, let K_ℓ^r be the complete r-graph on ℓ vertices. The problem of determining $\pi(K_\ell^r)$ was raised by Turán [37] and is still wide open. Erdős offered \$500 for the determination of any $\pi(K_\ell^r)$ with $\ell > r \geqslant 3$ and \$1000 for the determination of all $\pi(K_\ell^r)$ with $\ell > r \geqslant 3$.

Many families $\mathcal{F}$ have the property that there is a unique $\mathcal{F}$-free hypergraph $\mathcal{G}$ on n vertices achieving $\mathrm{ex}(n, \mathcal{F})$, and moreover, every $\mathcal{F}$-free hypergraph $\mathcal{H}$ of size close to $\mathrm{ex}(n, \mathcal{F})$ can be transformed to $\mathcal{G}$ by deleting and adding very few edges. Such a property is called *stability* of $\mathcal{F}$. The first stability theorem was proved independently by Erdős and Simonovits [36].

Theorem 1.1 (Erdős-Simonovits)**.** *Let $\ell \geqslant 2$ and let $\mathcal{F}$ be a family of graphs with $\chi(\mathcal{F}) = \ell + 1$. Then for every $\delta > 0$ there exist $\varepsilon > 0$ and $N_0 \in \mathbb{N}$ such that every $\mathcal{F}$-free graph on $n \geqslant N_0$ vertices with at least $(1 - \varepsilon)\mathrm{ex}(n, \mathcal{F})$ edges can be transformed to the Turán graph $T(n, \ell)$ by deleting and adding at most δn^2 edges.*   □

The stability phenomenon has been used to determine $\mathrm{ex}(n, \mathcal{F})$ exactly in many cases. It was first used by Simonovits in [36] to determine $\mathrm{ex}(n, F)$ exactly for all edge-critical graphs F and large n, and then by several authors (e.g. see [4, 13, 18, 19, 30, 32, 34]) to prove exact results for hypergraphs. In this article, stability will always mean stability relative to some intended class $\mathfrak{H}$ of 'almost extremal' $\mathcal{F}$-free graphs and we distinguish the following types of stability that have been studied in the literature.

Definition 1.2. *Let $\mathcal{F}$ be a nondegenerate family of r-graphs, where $r \geqslant 2$, and let $\mathfrak{H}$ be a class of $\mathcal{F}$-free r-graphs.*

- (*a*) *If for every $\delta > 0$ there exist $\varepsilon > 0$ and $N_0 \in \mathbb{N}$ such that every $\mathcal{F}$-free r-graph $\mathcal{H}$ on $n \geqslant N_0$ vertices with $|\mathcal{H}| \geqslant (\pi(\mathcal{F})/r! - \varepsilon)n^r$ becomes a subgraph of some member of $\mathfrak{H}$ after removing at most $\delta|\mathcal{H}|$ edges, then $\mathcal{F}$ is said to be edge-stable with respect to $\mathfrak{H}$.*
- (*b*) *If for every $\delta > 0$ there exist $\varepsilon > 0$ and $N_0 \in \mathbb{N}$ such that every $\mathcal{F}$-free r-graph $\mathcal{H}$ on $n \geqslant N_0$ vertices with $|\mathcal{H}| \geqslant (\pi(\mathcal{F})/r! - \varepsilon)n^r$ becomes a subgraph of some member of $\mathfrak{H}$ after removing at most $\delta|V(\mathcal{H})|$ vertices, then $\mathcal{F}$ is said to be vertex-stable with respect to $\mathfrak{H}$.*
- (*c*) *If there exist $\varepsilon > 0$ and N_0 such that every $\mathcal{F}$-free r-graph $\mathcal{H}$ on $n \geqslant N_0$ vertices with $\delta(\mathcal{H}) \geqslant \big(\pi(\mathcal{F})/(r-1)! - \varepsilon\big)n^{r-1}$ is a subgraph of some member of $\mathfrak{H}$ we say that $\mathcal{F}$ is degree-stable with respect to $\mathfrak{H}$.*

As a trivial example, every nondegenerate family $\mathcal{F}$ is stable in all three senses with respect to the class $\mathfrak{Forb}(\mathcal{F})$ of all $\mathcal{F}$-free r-graphs. More interestingly, Theorem 1.1 tells us that every family $\mathcal{F}$ of graphs with $\chi(F) = \ell + 1 \geqslant 3$ is edge-stable with respect to the class $\mathfrak{T}_\ell = \{T(n, \ell) \colon n \in \mathbb{N}\}$ of ℓ-partite Turán graphs.

In general, if a family $\mathcal{F}$ of r-graphs is degree-stable with respect to some class $\mathfrak{H}$, then a standard vertex deletion argument (see e.g. Fact 2.5 (a)) shows that $\mathcal{F}$ is vertex-stable with respect to $\mathfrak{H}$ as well. Moreover, since any $\delta v(\mathcal{H})$ vertices of an r-graph $\mathcal{H}$ cover at most $\delta v(\mathcal{H})^r$ edges of $\mathcal{H}$, it is in all interesting examples the case that if $\mathcal{F}$ is vertex-stable with respect to $\mathfrak{H}$, then it is edge-stable with respect to $\mathfrak{H}$ as well.

The goal of this work is to provide a unified framework for the stability of certain classes of graph and hypergraph families. Our main result (Theorem 1.7) reduces the stability of many problems to the much simpler task of checking that $\mathcal{F}$-free graphs or hypergraphs with large minimum degree have a property we call vertex-extendability (see Definition 1.6). The approach is designed for degree-stability and thus it not only simplifies the proofs of many known stability theorems but also gives stronger forms of these stability theorems.

1.2. **Main result.** Our results can be regarded as adding a new ingredient to Zykov's symmetrization method [39] and we commence by describing an 'axiomatic' framework for the determination of extremal numbers by means of symmetrization.

Given two r-graphs F and $\mathcal{H}$ a map $\varphi \colon V(F) \to V(\mathcal{H})$ is said to be a *homomorphism* if it preserves edges, i.e., if $\varphi(E) \in \mathcal{H}$ holds for all $E \in F$. If φ is surjective and every edge of $\mathcal{H}$ is an image of an edge of F, i.e., $\mathcal{H} = \{\varphi(E) \colon E \in F\}$, we call $\mathcal{H}$ a *homomorphic image* of F. Furthermore, $\mathcal{H}$ is F-*hom-free* if there is no homomorphism from F to $\mathcal{H}$. For a family $\mathcal{F}$ of r-graphs, we say that $\mathcal{H}$ is $\mathcal{F}$-*hom-free* if it is F-hom-free for every $F \in \mathcal{F}$. The forbidden families $\mathcal{F}$ studied in this article have the following property.

Definition 1.3 (Blowup-invariance). *A family $\mathcal{F}$ of r-graphs is* blowup-invariant *if every $\mathcal{F}$-free r-graph is $\mathcal{F}$-hom-free as well.*

For instance, for every $\ell \geqslant 2$ the families of graphs $\{K_\ell\}$ and $\{C_3, \ldots, C_{2\ell-1}\}$ are blowup invariant, whilst $\{C_5\}$ is not blowup-invariant. In the graph case one can easily check that a one-element family $\{F\}$ is blowup-invariant if and only if F is a clique, but for hypergraphs blowup-invariant families consisting of a single hypergraph F are much more common. In fact, if every pair of vertices of F is covered by an edge of F, then $\{F\}$ is blowup-invariant. One confirms easily that every family $\mathcal{F}$ closed under taking homomorphic images is blowup-invariant.

Let us now fix an r-graph $\mathcal{H}$. For every $v \in V(\mathcal{H})$ we call

$$L_\mathcal{H}(v) = \left\{ A \in \binom{V(\mathcal{H})}{r-1} \colon A \cup \{v\} \in \mathcal{H} \right\}$$

the *link* of v. Two vertices $u, v \in V(\mathcal{H})$ are said to be *equivalent* if $L_{\mathcal{H}}(u) = L_{\mathcal{H}}(v)$. Evidently, equivalence is an equivalence relation. We say that $\mathcal{H}$ is *symmetrized* if for any two non-equivalent vertices $u, v \in V(\mathcal{H})$ there is an edge $E \in \mathcal{H}$ containing both of them. For instance, a symmetrized graph is the same as a complete multipartite graph. We shall prove the following result by means of Zykov's symmetrization method.

Theorem 1.4. *Suppose that $\mathcal{F}$ is a blowup-invariant family of r-graphs. If $\mathfrak{H}$ denotes the class of all symmetrized $\mathcal{F}$-free r-graphs, then $\mathrm{ex}(n, \mathcal{F}) = \mathfrak{h}(n)$ holds for every $n \in \mathbb{N}$, where $\mathfrak{h}(n) = \max \{|\mathcal{H}|\colon \mathcal{H} \in \mathfrak{H} \text{ and } v(\mathcal{H}) = n\}$.*

Let us observe that this statement is very similar to the Lagrangian method developed and utilised by Motzkin-Straus [27], Sidorenko [35], Frankl-Füredi [11], and many others. Preparing the statement of our main result, we introduce some further notions. Recall that a class $\mathfrak{H}$ of r-graphs is called *hereditary* if it is closed under taking induced subgraphs.

Definition 1.5 (Symmetrized-stability)**.** *Let $\mathcal{F}$ be a family of r-graphs and let $\mathfrak{H}$ be a class of $\mathcal{F}$-free r-graphs. We say that $\mathcal{F}$ is symmetrized-stable with respect to $\mathfrak{H}$ if there exist $\varepsilon > 0$ and N_0 such that every symmetrized $\mathcal{F}$-free r-graphs $\mathcal{H}$ on $n \geqslant N_0$ vertices with $|\mathcal{H}| \geqslant (\pi(\mathcal{F})/r! - \varepsilon)n^r$ is a subgraph of a member of $\mathfrak{H}$.*

The next definition might be the most important one in this article.

Definition 1.6 (Vertex-extendibility)**.** *Let $\mathcal{F}$ be a family of r-graphs and let $\mathcal{H}$ be a class of $\mathcal{F}$-free r-graphs. We say that $\mathcal{F}$ is vertex-extendable with respect to $\mathfrak{H}$ if there exist $\zeta > 0$ and $N_0 \in \mathbb{N}$ such that for every $\mathcal{F}$-free r-graph $\mathcal{H}$ on $n \geqslant N_0$ vertices satisfying $\delta(\mathcal{H}) \geqslant \big(\pi(\mathcal{F})/(r-1)! - \zeta\big)n^{r-1}$ the following holds: if $\mathcal{H} - v$ is a subgraph of a member of $\mathfrak{H}$ for some vertex $v \in V(\mathcal{H})$, then $\mathcal{H}$ is a subgraph of a member of $\mathfrak{H}$ as well.*

We can now state our sufficient conditions for degree-stability.

Theorem 1.7 (Main result)**.** *Suppose that $\mathcal{F}$ is a blowup-invariant nondegenerate family of r-graphs and that $\mathfrak{H}$ is a hereditary class of $\mathcal{F}$-free r-graphs. If $\mathcal{F}$ is symmetrized-stable and vertex-extendable with respect to $\mathfrak{H}$, then $\mathcal{F}$ is degree-stable with respect to $\mathfrak{H}$ as well.*

In practice the assumptions on $\mathfrak{H}$ are often easy to verify but it may happen that the family $\mathcal{F}$ we want to study fails to be blowup-invariant. If in such a situation we know for any reason that $\mathcal{F}$ is vertex-stable with respect to $\mathfrak{H}$, we can improve this information to degree-stability.

Theorem 1.8. *Suppose that $\mathcal{F}$ is a nondegenerate family of r-graphs and that $\mathcal{H}$ is a hereditary class of $\mathcal{F}$-free r-graphs. If $\mathcal{F}$ is vertex-stable and vertex-extendable with respect to $\mathfrak{H}$, then it is degree-stable with respect to $\mathfrak{H}$ as well.*

1.3. **Further results and applications.** An r-graph $\mathcal{H}$ is said to be a *blowup* of another r-graph F is there exists a map $\psi\colon V(\mathcal{H}) \longrightarrow V(F)$ such every $E \in \binom{V(\mathcal{H})}{r}$ satisfies the equivalence $\psi(E) \in F \Longleftrightarrow E \in \mathcal{H}$. If ψ is surjective, the blowup is called *proper*. Subgraphs of blowups of F are called *F-colorable*.

For integers $\ell \geqslant r \geqslant 2$ let $\mathfrak{K}_\ell^r$ be the class of all blowups of K_ℓ^r. If $r = 2$ we omit the superscript and just write $\mathfrak{K}_\ell$ for the class of complete ℓ-partite graphs (whose vertex classes are allowed to be empty). Most but not all stability results described below are with respect to classes of the form $\mathfrak{K}_\ell^r$.

1.3.1. *Graphs.* The classical stability theorem of Erdős and Simonovits (Theorem 1.1) informs us that every family $\mathcal{F}$ of graphs with $\chi(\mathcal{F}) = \ell + 1 \geqslant 3$ is edge-stable with respect to $\mathfrak{K}_\ell$. Complementing this result one can also characterise the families of graphs which are degree-stable and vertex-stable with respect to $\mathfrak{K}_\ell$. To this end we recall that a graph F is said to be *edge-critical* if it has an edge $e \in F$ such that $\chi(F - e) < \chi(F)$ and *matching-critical* if there exists a matching $M \subseteq F$ such that $\chi(F - M) < \chi(F)$. More generally, we call a family $\mathcal{F}$ of graphs *edge-critical* or *matching-critical* if there exists a graph $F \in \mathcal{F}$ with $\chi(F) = \chi(\mathcal{F})$ that is edge-critical or matching-critical. In the result that follows, part (b) is due to Erdős and Simonovits [9], while part (a) might very well be new.

Theorem 1.9. *A family $\mathcal{F}$ of graphs with $\chi(\mathcal{F}) = \ell + 1 \geqslant 3$ is*

(a) *vertex-stable with respect to $\mathfrak{K}_\ell$ if and only if it is matching-critical*

(b) *and degree-stable with respect to $\mathfrak{K}_\ell$ if and only if it is edge-critical.*

1.3.2. *Cancellative hypergraphs and generalized triangles.* An r-graph $\mathcal{H}$ is *cancellative* if $A \cup B = A \cup C$ implies that $B = C$ for all $A, B, C \in \mathcal{H}$. Since $A \cup B = A \cup C$ is equivalent to $B \triangle C \subseteq A$, an r-graph $\mathcal{H}$ is cancellative if and only if it is $\mathcal{T}_r$-free, where $\mathcal{T}_r$ denotes the family consisting of all r-graphs with three edges one of which contains the symmetric difference of the two other ones.

It was conjectured by Katona and proved by Bollobás [3] that the maximum number of edges in an n-vertex $\mathcal{T}_3$-free 3-graph is uniquely achieved by the balanced complete 3-partite 3-graph. Keevash and the second author [17] proved that $\mathcal{T}_3$ is edge-stable with respect to $\mathfrak{K}_3^3$, and the first author [20] discovered another short proof of the edge-stability of $\mathcal{T}_3$ giving a linear dependency between the error parameters. Sidorenko [35] proved that the maximum number of edges in an n-vertex $\mathcal{T}_4$-free 4-graph is uniquely achieved by the balanced complete 4-partite 4-graph. Later, Pikhurko [33] proved that $\mathcal{T}_4$ is vertex-stable with respect to $\mathfrak{K}_4^4$ using a sophisticated variation of Zykov symmetrization. For $r \geqslant 5$ the value of $\pi(\mathcal{T}_r)$ is unknown.

Cancellative hypergraphs are closely related to the Turán problem for generalized triangles. For $r \geqslant 2$ let Σ_r be the collection of all r-graphs with three edges A, B, C such that $|B \cap C| = r - 1$ and $B \triangle C \subseteq A$. The unique r-graph $T_r \in \Sigma_r$ with $v(T_r) = 2r - 1$ is called the *generalized triangle*. It is easy to see that $\Sigma_2 = \mathcal{T}_2 = \{K_3\}$, $\Sigma_3 = \mathcal{T}_3$, and $\Sigma_r \subsetneq \mathcal{T}_r$ for $r \geqslant 4$.

The results on $\mathcal{T}_4$ due to Sidorenko [35] and Pikhurko [33] quoted earlier hold for Σ_4 instead of $\mathcal{T}_4$ as well. In particular, Σ_4 is known to be vertex-stable with respect to $\mathfrak{K}_4^4$. For $r = 5, 6$ Frankl and Füredi [11] proved that the extremal numbers $\mathrm{ex}(n, \Sigma_r)$ are only realized by balanced blowups of the famous Witt designs [38] with parameters $(11, 5, 4)$ and $(12, 6, 5)$, respectively. Norin and Yepremyan [31] proved that Σ_5 and Σ_6 are edge-stable with respect to blowups of these Witt-designs, but Pikhurko showed [33] that they fail to be vertex-stable. For $r \geqslant 7$ it is an open problem to determine $\pi(\Sigma_r)$.

Theorem 1.10. *For $r \in \{3, 4\}$ the family Σ_r is degree-stable with respect to $\mathfrak{K}_r^r$.*

1.3.3. *Hypergraph expansions.* Given an r-graph F and $i \in [r - 1]$ we write $\partial_i F$ for the $(r - i)$-graph with the same vertex set as F whose edges are the $(r - i)$-subsets of $V(F)$ covered by an edge of F. In particular, $\partial_{r-2} F$ is a graph on $V(F)$. A set $X \subseteq V(F)$ is called 2-*covered in F* if it induces a clique in $\partial_{r-2} F$. If $V(F)$ itself is 2-covered in F we simply say that F is 2-*covered*. The *neighborhood* $N_F(v)$ of a vertex $v \in V(F)$ is defined to be the set of all $u \in V(F) \smallsetminus \{v\}$ with $\{u, v\} \in \partial_{r-2} F$.

For an r-graph F with ℓ vertices we define $\mathcal{K}_\ell^F$ to be the set of all r-graphs of the form $F \cup \{S_{uv} : uv \in \binom{V(F)}{2} \smallsetminus \partial_{r-2} F\}$, where for every pair of vertices $uv \in \binom{V(F)}{2} \smallsetminus \partial_{r-2} F$ not covered by an edge of F the edge S_{uv} contains u and v. We write H_ℓ^F for the unique member of $\mathcal{K}_\ell^F$ having the largest number of vertices, namely

$$v(H_\ell^F) = \ell + (r - 2)\left(\binom{\ell}{2} - |\partial_{r-2} F|\right).$$

The r-graphs in $\mathcal{K}_\ell^F$ are called *weak expansions* of F while H_ℓ^F is called the *expansion* of F. If F has no edges (and thus consists of ℓ isolated vertices) we write $\mathcal{K}_\ell^r$ and H_ℓ^r instead of $\mathcal{K}_\ell^F$ and H_ℓ^F.

The notion of hypergraph expansions was first introduced by the second author in [28] to extend Turán's theorem to hypergraphs. In [28] it was proved that for every $n \geqslant \ell \geqslant r \geqslant 2$ the maximum number of edges in an n-vertex $\mathcal{K}_{\ell+1}^r$-free r-graph is uniquely achieved by $T_r(n, \ell)$, the balanced complete ℓ-partite r-graph on n vertices. In addition, [28] proved that $\mathcal{K}_{\ell+1}^r$ is edge-stable with respect to $\mathcal{K}_{\ell+1}^r$. Later de Oliveira Contiero, Hoppen, Lefmann, and Odermann [6], and independently, the first author [20] improved the edge-stability result by showing that a linear dependence between δ and ε is sufficient. Pikhurko [34]

refined [28] by showing that $T_r(n, \ell)$ is also the unique $H_{\ell+1}^r$-free r-graph on n vertices with the maximum number of edges for sufficiently large n.

Keevash [16] observed a generalization of these results to expansions of a large class of hypergraphs F. Let us write $\lambda(\mathcal{G})$ for the Lagrangian of a hypergraph $\mathcal{G}$ (see Section 2 for the definition) and set $\pi_\lambda(F) = \sup\{\lambda(\mathcal{G}) \colon \mathcal{G} \text{ is } F\text{-free}\}$ for every r-graph F.

Theorem 1.11 (Keevash)**.** *Let F be an r-graph with $v(F) = \ell + 1$. If $\pi_\lambda(F) \leq \binom{\ell}{r}/\ell^r$, then*

$$\mathrm{ex}(n, \mathcal{K}_{\ell+1}^F) \leq \binom{\ell}{r} n^r/\ell^r$$

holds for all positive integers n, and equality occurs whenever n is divisible by r. In particular,

$$\pi(H_{\ell+1}^F) = \pi(\mathcal{K}_{\ell+1}^F) = (\ell)_r/\ell^r.$$

In the special case where F has an isolated vertex and $\pi_\lambda(F) < \binom{\ell}{r}/\ell^r$ Brandt, Irwin, and Jiang [4], and independently, Norin and Yepremyan [32] proved a stability theorem for the family $\mathcal{K}_{\ell+1}^F$ and used it to determine $\mathrm{ex}(n, H_\ell^r)$ exactly for all sufficiently large integers n. More specifically, [4] shows that $\mathcal{K}_{\ell+1}^F$ is vertex-stable, and [32] shows that $\mathcal{K}_{\ell+1}^F$ is edge-stable. Our result below shows the stronger fact that $\mathcal{K}_{\ell+1}^F$ is degree-stable.

Moreover, we prove degree-stability in many cases where F has no isolated vertices but is contained instead in the hypergraph $B(r, \ell + 1)$ with vertex set $[\ell + 1]$ and edge set

$$\{[r]\} \cup \{E \subseteq [2, \ell + 1] \colon |E| = r \text{ and } |[2, r] \cap E| \leq 1\}.$$

Theorem 1.12. *Let $\ell \geq r \geq 2$ and suppose that F is an r-graph satisfying $v(F) = \ell + 1$ and*

$$\sup\{\lambda(\mathcal{G}) \colon \mathcal{G} \text{ is } F\text{-free and not } K_\ell^r\text{-colorable}\} < \binom{\ell}{r}/\ell^r. \tag{1.1}$$

If either F has an isolated vertex or $F \subseteq B(r, \ell + 1)$, then the family $\mathcal{K}_{\ell+1}^F$ is degree-stable with respect to $\mathfrak{K}_\ell^r$.

There are several natural examples of hypergraphs F which have been proved to satisfy condition (1.1) in the literature but whose families of weak expansions $\mathcal{K}_{\ell+1}^F$ were not known to be degree-stable before. For instance, Hefetz and Keevash [14] studied the case that $F = M_2^3$ is a 3-uniform matching with two edges and six vertices. More generally Jiang, Peng, and Wu [15] proved the assumption if $F = \{M_t^3, L_t^3, L_t^4\}$ holds for some $t \geq 2$; here M_t^3 denotes the 3-uniform matching of size t and for $r \geq 2$ the r-graph L_t^r consists of t edges having one vertex v in common such that $E \cap E' = \{v\}$ holds for all distinct $E, E' \in L_t^r$. Brandt, Irwin, and Jiang [4] proved that in these cases the families $\mathcal{K}^F$ are vertex-stable. By combining the results in [15] on Lagrangians with Theorem 1.12 one immediately obtains the following strengthening of this fact.

Corollary 1.13. *For $t \geqslant 2$ the families $\mathcal{K}_{3t}^{M_t^3}$, $\mathcal{K}_{2t+1}^{L_t^3}$, $\mathcal{K}_{3t+1}^{L_t^4}$ are degree-stable with respect to $\mathfrak{K}_{3t-1}^3$, $\mathfrak{K}_{2t}^3$, $\mathfrak{K}_{3t}^4$, respectively.* $\square$

1.3.4. *Expansions of matchings of size 2.* For $r \geqslant 3$ let M_2^r be the r-graph on $2r$ vertices consisting of two disjoint edges. The trivial observation that no r-graph in $\mathfrak{K}_{2r-1}^r$ contains a weak expansion of M_2^r yields the lower bound $\pi(\mathcal{K}_{2r}^{M_2^r}) \geqslant (2r-1)_r/(2r-1)^r$. In their work [14] establishing equality for $r = 3$ Hefetz and Keevash also observed that for $r \geqslant 4$ there is a denser construction of $\mathcal{K}_{2r}^{M_2^r}$-free r-graphs.

Call an r-graph $\mathcal{H}$ *semibipartite* if there exists a partition $V(\mathcal{H}) = A \uplus B$ such that $|A \cap E| = 1$ holds for every $E \in \mathcal{H}$. If $\mathcal{H}$ contains all $|A|\binom{|B|}{r-1}$ such edges we say that $\mathcal{H}$ is a *complete semibipartite r-graph*. It is easy to see that semibipartite r-graphs cannot contain weak expansions of M_2^r and that $(1 - 1/r)^{r-1}$ is the supremum of the edge densities of semibipartite r-graphs. A straightforward calculation shows that for $r \geqslant 4$ this number is indeed larger than the lower bound $(2r-1)_r/(2r-1)^r$ mentioned before. In fact Hefetz and Keevash [14] conjectured $\pi(\mathcal{K}_{2r}^{M_2^r}) = (1 - 1/r)^{r-1}$ for every $r \geqslant 4$. This was proved by Bene Watts, Norin, and Yepremyan [2] who also showed that $\mathcal{K}_{2r}^{M_2^r}$ is edge-stable with respect to the class $\mathfrak{S}^r$ of all complete semibipartite r-graphs. Combining a substantial result on Lagrangians from their work with our Theorem 1.7 we strengthen this to degree-stability.

Theorem 1.14. *For every $r \geqslant 4$ the family of weak expansions of M_2^r is degree-stable with respect to $\mathfrak{S}^r$.*

Organization. In Section 2 we introduce some definitions and useful lemmas. The results presented in Subsection 1.2 are proved in Section 3 and Section 4 deals with the applications described in Subsection 1.3. The last section consists of concluding remarks.

§2. PRELIMINARIES

We begin with some definitions related to the Lagrangians of hypergraphs (introduced by Frankl-Rödl [12]). Given an r-graph $\mathcal{G}$ on m vertices (let us assume for notational transparency that $V(\mathcal{G}) = [m]$) the multilinear function $L_{\mathcal{G}} \colon \mathbb{R}^m \to \mathbb{R}$ is defined by

$$L_{\mathcal{G}}(x_1, \ldots, x_m) = \sum_{E \in \mathcal{H}} \prod_{i \in E} x_i, \quad \text{for all } (x_1, \ldots, x_m) \in \mathbb{R}^m.$$

Denote by Δ_{m-1} the standard $(m-1)$-dimensional simplex, i.e.

$$\Delta_{m-1} = \{(x_1, \ldots, x_m) \in [0, 1]^n \colon x_1 + \cdots + x_m = 1\}.$$

Since Δ_{m-1} is compact, a theorem of Weierstraß implies that the restriction of $L_{\mathcal{G}}$ to Δ_{m-1} attains a maximum value, called the *Lagrangian* of $\mathcal{G}$ and denoted by $\lambda(\mathcal{G})$.

Lagrangians arise naturally when one considers the maximum possible densities of blowups. Given an r-graph $\mathcal{G}$ with vertex set $[m]$ and mutually disjoint sets $V_1, \ldots, V_m$ we

write $\mathcal{G}[V_1, \ldots, V_m]$ for the r-graph obtained from $\mathcal{G}$ upon replacing every vertex $i \in [m]$ by the set V_i and every edge $\{i(1), \ldots, i(r)\} \in \mathcal{G}$ by the complete r-partite r-graph with vertex classes $V_{i(1)}, \ldots, V_{i(r)}$. Obviously we have

$$|\mathcal{G}[V_1, \ldots, V_m]| = L_{\mathcal{G}}(|V_1|, \ldots, |V_m|) \tag{2.1}$$

in this situation. If $\mathcal{H}$ is a spanning subgraph of $\mathcal{G}[V_1, \ldots, V_m]$ we say that the partition $V(\mathcal{H}) = \bigcup_{i \in [m]} V_i$ is a $\mathcal{G}$-*coloring* of $\mathcal{H}$. Observe that an r-graph $\mathcal{H}$ admits a $\mathcal{G}$-coloring if and only if it is $\mathcal{G}$-colorable. Since $L_{\mathcal{G}}$ is a homogeneous polynomial of degree r, the formula (2.1) immediately implies the following observation (e.g. see [11, 16]).

Lemma 2.1. *Let $\mathcal{G}$ be an r-graph on m vertices. If $\mathcal{H}$ denotes an r-graph on n vertices possessing a $\mathcal{G}$-coloring $V(\mathcal{H}) = \bigcup_{i \in [m]} V_i$ and $x_i = |V_i|/n$ for every $i \in [m]$, then*

$$|\mathcal{H}| \leqslant L_{\mathcal{G}}(x_1, \ldots, x_m)n^r \leqslant \lambda(\mathcal{G})n^r. \qquad \square$$

The next result, concerning the Lagrangian of the complete r-graph K_m^r, will be useful in proofs of stability theorems whose extremal configuration is a blowup of K_m^r.

Lemma 2.2. *If $m \geqslant r \geqslant 2$ and $(x_1, \ldots, x_m) \in \Delta_{m-1}$, then*

$$L_{K_m^r}(x_1, \ldots, x_m) + \frac{\binom{m}{r}}{m^{r-1}(m-1)} \sum_{i \in [m]} \left(x_i - \frac{1}{m}\right)^2 \leqslant \frac{1}{m^r}\binom{m}{r}.$$

Clearly this holds with equality if $(x_1, \ldots, x_m)$ is either $(1/m, \ldots, 1/m)$ or a unit vector.

Proof of Lemma 2.2. Setting $\mu_i = L_{K_m^i}(x_1, \ldots, x_m)$ for every $i \in [m]$, we have $\mu_1 = 1$ and Maclaurin's inequality implies

$$\left(\frac{\mu_r}{\binom{m}{r}}\right)^{1/r} \leqslant \cdots \leqslant \left(\frac{\mu_2}{\binom{m}{2}}\right)^{1/2} \leqslant \frac{\mu_1}{\binom{m}{1}} = \frac{1}{m}.$$

Consequently,

$$\frac{\mu_2}{\binom{m}{2}} \geqslant \left(\frac{\mu_r}{\binom{m}{r}}\right)^{2/r} \geqslant \left(\frac{\mu_r}{\binom{m}{r}}\right)^{2/r} \cdot \left(m^r \frac{\mu_r}{\binom{m}{r}}\right)^{(r-2)/r},$$

i.e.,

$$\mu_r \leqslant \frac{\binom{m}{r}}{m^{r-2}} \cdot \frac{\mu_2}{\binom{m}{2}}.$$

Since

$$2\mu_2 = 1 - \sum_{i \in [m]} x_i^2 = \frac{m-1}{m} - \sum_{i \in [m]} \left(x_i - \frac{1}{m}\right)^2,$$

the result follows. $\qquad \square$

Easy calculations based on Lemma 2.2 imply the following.

Corollary 2.3. *Given $m \geqslant r \geqslant 2$ let $\zeta > 0$ be sufficiently small. Suppose further that $\mathcal{H}$ is an n-vertex r-graph admitting a K_m^r-coloring $V(\mathcal{H}) = \bigcup_{i \in [m]} V_i$.*

  (a) If $|\mathcal{H}| \geqslant \left(\binom{m}{r}/m^r - \zeta\right) n^r$, then $|V_i| = (1/m \pm C_1\zeta^{1/2})n$ holds for every $i \in [m]$, where $C_1 = \left(m^{r-1}(m-1)/\binom{m}{r}\right)^{1/2}$.

  (b) If $\delta(\mathcal{H}) \geqslant \left(\binom{m-1}{r-1}/m^{r-1} - \zeta\right)n^{r-1}$ and $v \in V_i$, then $|V_j \smallsetminus N_{\mathcal{H}}(v)| \leqslant 2C_1\zeta^{1/2}n$ holds for every $j \in [m] \smallsetminus \{i\}$.

  (c) If $\delta(\mathcal{H}) \geqslant \left(\binom{m-1}{r-1}/m^{r-1} - \zeta\right)n^{r-1}$, then $\left|L_{\widehat{K}_m^r}(v) \smallsetminus L_{\mathcal{H}}(v)\right| \leqslant C_2\zeta^{1/2}n^{r-1}$ holds for every $v \in V(\mathcal{H})$, where $\widehat{K}_m^r = K_m^r[V_1, \ldots, V_m]$ and $C_2 = r\binom{m-1}{r-1}C_1/m^{r-2}$.   □

The special case $r = 3$ of the following lemma was stated in [24, Lemma 4.5]. It is straightforward to generalize the probabilistic proof provided there to the general case and we omit the details.

Lemma 2.4. *Fix a real $\eta \in (0, 1)$ and integers $m, n \geqslant 1$, $r \geqslant 3$. Let $\mathcal{G}$ be an r-graph with vertex set $[m]$ and let $\mathcal{H}$ be an r-graph with $v(\mathcal{H}) = n$. Consider a vertex partition $V(\mathcal{H}) = \bigcup_{i \in [m]} V_i$ and the associated blowup $\widehat{\mathcal{G}} = \mathcal{G}[V_1, \ldots, V_m]$ of $\mathcal{G}$. If two sets $T \subseteq [m]$ and $S \subseteq \bigcup_{j \notin T} V_j$ have the properties*

  (a) $|V_j| \geqslant (|S| + 1)|T|\eta^{1/r}n$ for all $j \in T$,
  (b) $|\mathcal{H}[V_{j_1}, \ldots, V_{j_r}]| \geqslant |\widehat{\mathcal{G}}[V_{j_1}, \ldots, V_{j_r}]| - \eta n^r$ for all $\{j_1, \ldots, j_r\} \in \binom{T}{r}$,
  (c) and $|L_{\mathcal{H}}(v)[V_{j_1}, \ldots, V_{j_{r-1}}]| \geqslant |L_{\widehat{\mathcal{G}}}(v)[V_{j_1}, \ldots, V_{j_{r-1}}]| - \eta n^{r-1}$ for all $v \in S$ and all $\{j_1, \ldots, j_{r-1}\} \in \binom{T}{r-1}$,

then there exists a selection of vertices $u_j \in V_j$ for all $j \in [T]$ such that $U = \{u_j : j \in T\}$ satisfies $\widehat{\mathcal{G}}[U] \subseteq \mathcal{H}[U]$ and $L_{\widehat{\mathcal{G}}}(v)[U] \subseteq L_{\mathcal{H}}(v)[U]$ for all $v \in S$. In particular, if $\mathcal{H} \subseteq \widehat{\mathcal{G}}$, then $\widehat{\mathcal{G}}[U] = \mathcal{H}[U]$ and $L_{\widehat{\mathcal{G}}}(v)[U] = L_{\mathcal{H}}(v)[U]$ for all $v \in S$.   □

For the proof of the following standard fact we refer to [24, Lemma 4.2].

Fact 2.5. *Let $\mathcal{F}$ be a family of r-graphs and let $\mathcal{H}$ be an $\mathcal{F}$-free r-graph on n vertices. If $\mathcal{H}$ has at least $(\pi(\mathcal{F})/r! - \varepsilon)n^r$ edges, then*

  (a) the set $Z_\varepsilon(\mathcal{H}) = \left\{u \in V(\mathcal{H}) : d_{\mathcal{H}}(u) \leqslant \left(\pi(\mathcal{F})/(r-1)! - 2\varepsilon^{1/2}\right)n^{r-1}\right\}$ has size at most $\varepsilon^{1/2}n$,
  (b) and the r-graph $\mathcal{H}' = \mathcal{H} - Z_\varepsilon(\mathcal{H})$ satisfies $\delta(\mathcal{H}') > \left(\pi(\mathcal{F})/(r-1)! - 3\varepsilon^{1/2}\right)n^{r-1}$.   □

§3. PROOF OF THE MAIN RESULT: THE Ψ-TRICK

We prove Theorems 1.4, 1.7, and 1.8 in this section. To this end the following pieces of notation will be convenient: If C denotes an equivalence class of some hypergraph $\mathcal{H}$, we write $d_{\mathcal{H}}(C)$ for the common degree of the vertices in C and $L_{\mathcal{H}}(C)$ for their common link.

Given a class of hypergraphs $\mathfrak{H}$ we denote the class of spanning subgraphs of members of $\mathfrak{H}$ by $\mathfrak{H}^+$, i.e. we set

$$\mathfrak{H}^+ = \{\mathcal{H} \colon \text{there is } \mathcal{H}' \in \mathfrak{H} \text{ with } V(\mathcal{H}) = V(\mathcal{H}') \text{ and } \mathcal{H} \subseteq \mathcal{H}'\}.$$

If $\mathfrak{H}$ is hereditary, this is the same as the class of (not necessarily spanning) subgraphs of members of $\mathfrak{H}$.

Proof of Theorem 1.4. Fix $n \in \mathbb{N}$. The lower bound $\mathrm{ex}(n, \mathcal{F}) \geqslant \mathfrak{h}(n)$ is an immediate consequence of the fact that all members of $\mathfrak{H}$ are $\mathcal{F}$-free. So it remains to establish the upper bound $\mathrm{ex}(n, \mathcal{F}) \leqslant \mathfrak{h}(n)$.

Suppose that it is not true and let $\mathcal{H}$ be an $\mathcal{F}$-free r-graph on n vertices with more than $\mathfrak{h}(n)$ edges chosen in such a way that the number m of its equivalence classes is minimum. Let $C_1, \ldots, C_m$ denote the equivalence classes of $\mathcal{H}$.

Due to $|\mathcal{H}| > \mathfrak{h}(n)$ we know that $\mathcal{H}$ cannot be symmetrized. In other words, there exist $i, j \in [m]$ such that the graph $\partial_{r-2}\mathcal{H}$ is not complete between C_i and C_j. Without loss of generality we may assume that $\{i, j\} = \{1, 2\}$ and $d_{\mathcal{H}}(C_1) \leqslant d_{\mathcal{H}}(C_2)$. In view of the definition of equivalence there are actually no edges between C_1 and C_2 in $\partial_{r-2}\mathcal{H}$.

Now let $\mathcal{H}'$ be the unique r-graph satisfying $V(\mathcal{H}') = V(\mathcal{H})$, $\mathcal{H}' - C_1 = \mathcal{H} - C_1$, and $L_{\mathcal{H}'}(v) = L_{\mathcal{H}}(w)$ for all $v \in C_1$ and $w \in C_2$. Observe that $\{C_1 \cup C_2, C_3, \ldots, C_m\}$ is a refinement of the partition of $V(\mathcal{H}')$ into equivalence classes of $\mathcal{H}'$, for which reason $\mathcal{H}'$ has fewer than m equivalence classes. Together with

$$|\mathcal{H}'| = |\mathcal{H}| + |C_1|\left(d_{\mathcal{H}}(C_2) - d_{\mathcal{H}}(C_1)\right) \geqslant |\mathcal{H}| > \mathfrak{h}(n)$$

and our minimal choice of m this implies that $\mathcal{H}'$ cannot be $\mathcal{F}$-free. As there exists a homomorphism from $\mathcal{H}'$ to $\mathcal{H}$, it follows that $\mathcal{H}$ fails to be $\mathcal{F}$-hom-free. But, as $\mathcal{F}$ is blowup-invariant, this contradicts the assumption that $\mathcal{H}$ be $\mathcal{F}$-free. $\qquad\square$

For the proof of Theorem 1.7 it will be convenient to say for $\zeta > 0$ and $N_0 \in \mathbb{N}$ that a family $\mathcal{F}$ of r-graphs is (ζ, N_0)-vertex-extendable with respect to a class of r-graphs $\mathfrak{H}$ if using the notation of Definiton 1.6 ζ and N_0 exemplify the vertex-extendibility of $\mathcal{F}$ with respect to $\mathfrak{H}$. The next lemma shows that vertex-extendibility can be used iteratively.

Lemma 3.1. *Suppose that the nondegenerate family $\mathcal{F}$ of r-graphs is $(2\varepsilon, N_0)$-vertex-extendable with respect to a class $\mathfrak{H}$ of $\mathcal{F}$-free r-graphs, where $\varepsilon \in (0, 1/2)$ and $N_0 \in \mathbb{N}$. Let $\mathcal{H}$ be an $\mathcal{F}$-free r-graph on $n \geqslant 2N_0$ vertices satisfying $\delta(\mathcal{H}) \geqslant \left(\pi(\mathcal{F})/(r-1)! - \varepsilon\right)n^{r-1}$. If there exists a set $S \subseteq V(\mathcal{H})$ with $|S| \leqslant \varepsilon n$ and $(\mathcal{H} - S) \in \mathfrak{H}^+$, then $\mathcal{H} \in \mathfrak{H}^+$.*

Proof of Lemma 3.1. Choose a minimal set $S' \subseteq S$ with $(\mathcal{H} - S') \in \mathfrak{H}^+$. If $S' = \varnothing$ we are done, so suppose for the sake of contradiction that there exists a vertex $v \in S'$. Setting

$S'' = S' \smallsetminus \{v\}$ and $\mathcal{H}'' = \mathcal{H} - S''$ we have $v(\mathcal{H}'') \geqslant n - |S| \geqslant (1-\varepsilon)n \geqslant n/2 \geqslant N_0$ and

$$\delta(\mathcal{H}'') \geqslant \delta(\mathcal{H}) - |S''|n^{r-2} > \big(\pi(\mathcal{F})/(r-1)! - \varepsilon\big)n^{r-1} - \varepsilon n^{r-1}$$
$$\geqslant \big(\pi(\mathcal{F})/(r-1)! - 2\varepsilon\big)v(\mathcal{H}'')^{r-1}.$$

Moreover, we are assuming that $\mathcal{H}'' - v = \mathcal{H} - S'$ is in $\mathfrak{H}^+$. So by vertex-extendibility $\mathcal{H}''$ belongs to $\mathfrak{H}^+$ as well and S'' contradicts the minimality of S'. $\qquad\square$

Next we shall show the following strengthening of Theorem 1.7 which also allows vertices of low degree in the almost extremal $\mathcal{F}$-free graphs. Recall that the sets $Z_\varepsilon(\mathcal{H})$ appearing below were defined in Fact 2.5 (a).

Theorem 3.2. *Let $\mathcal{F}$ be a blowup-invariant nondegenerate family of r-graphs and let $\mathfrak{H}$ be a hereditary class of $\mathcal{F}$-free r-graphs. If $\mathcal{F}$ is symmetrized-stable and vertex-extendable with respect to $\mathfrak{H}$, then there are $\varepsilon > 0$ and $N_0 \in \mathbb{N}$ such that every $\mathcal{F}$-free r-graph $\mathcal{H}$ on $n \geqslant N_0$ vertices with $|\mathcal{H}| > (\pi(\mathcal{F})/r! - \varepsilon)\, n^r$ satisfies $\mathcal{H} - Z_\varepsilon(\mathcal{H}) \in \mathfrak{H}^+$.*

The proof involves the following invariant of hypergraphs: If $C_1, \ldots, C_m$ are the equivalence classes of an r-graph $\mathcal{H}$, we set $\Psi(\mathcal{H}) = \sum_i |C_i|^2$.

Proof of Theorem 3.2. Choose $\varepsilon \in (0, 1/36)$ so small and $N_0 \in \mathbb{N}$ so large that

(1) the symmetrized stability of $\mathcal{F}$ with respect to $\mathfrak{H}$ is exemplified by ε and N_0

(2) and $\mathcal{F}$ is $(6\varepsilon^{1/2}, N_0/3)$-vertex-extendable with respect to $\mathfrak{H}$.

Now we fix $n \geqslant N_0$ and, assuming that the conclusion fails for n-vertex hypergraphs $\mathcal{H}$, we pick a counterexample $\mathcal{H}$ with $v(\mathcal{H}) = n$ such that the pair $(|\mathcal{H}|, \Psi(\mathcal{H}))$ is lexicographically maximal (which makes sense, as n is fixed). Let $C_1, \ldots, C_m$ be the equivalence classes of $\mathcal{H}$.

Setting $Z = Z_\varepsilon(\mathcal{H})$ we have $(\mathcal{H} - Z) \notin \mathfrak{H}^+$ and, as $\mathfrak{H}$ is hereditary, $\mathcal{H} \notin \mathfrak{H}^+$ follows. Now (1) informs us that $\mathcal{H}$ is not symmetrized. So without loss of generality we may suppose that $\partial_{r-2}\mathcal{H}$ has no edges between C_1 and C_2 and that $(d_\mathcal{H}(C_1), |C_1|) \leqslant_{\text{lex}} (d_\mathcal{H}(C_2), |C_2|)$, where $\leqslant_{\text{lex}}$ means lexicographic ordering.

Now we pick two arbitrary vertices $v_1 \in C_1$ and $v_2 \in C_2$ and symmetrize only them, i.e., we let $\mathcal{H}'$ be the r-graph with $V(\mathcal{H}') = V(\mathcal{H})$, $\mathcal{H}' - v_1 = \mathcal{H} - v_1$ and $L_{\mathcal{H}'}(v_1) = L_\mathcal{H}(v_2)$. Clearly, if $d_\mathcal{H}(v_1) < d_\mathcal{H}(v_2)$, then $|\mathcal{H}'| > |\mathcal{H}|$. Moreover, if $d_\mathcal{H}(v_1) = d_\mathcal{H}(v_2)$, then $|\mathcal{H}'| = |\mathcal{H}|$, $|C_1| \leqslant |C_2|$, and

$$\Psi(\mathcal{H}') - \Psi(\mathcal{H}) \geqslant (|C_1| - 1)^2 + (|C_2| + 1)^2 - |C_1|^2 - |C_2|^2 = 2(|C_2| - |C_1| + 1) \geqslant 2.$$

In both cases $(|\mathcal{H}'|, \Psi(\mathcal{H}'))$ is lexicographically larger than $(|\mathcal{H}|, \Psi(\mathcal{H}))$ and our choice of $\mathcal{H}$ implies $\mathcal{H}' - Z_\varepsilon(\mathcal{H}') \in \mathfrak{H}^+$. By Fact 2.5 (a) the set $Q = Z_\varepsilon(\mathcal{H}') \cup \{v_1\}$ satisfies $|Q| \leqslant \varepsilon^{1/2}n + 1 < 2\varepsilon^{1/2}n$. Since $\mathfrak{H}$ is hereditary, the r-graph $\mathcal{H} - Q = \mathcal{H}' - Q$ belongs to $\mathfrak{H}^+$.

Now Fact 2.5 (b) and (2) show that Lemma 3.1 applies to $3\varepsilon^{1/2}$, $2N_0/3$, $\mathcal{H} - (Q \cap Z)$, and $Q \smallsetminus Z$ here in place of ε, N_0, $\mathcal{H}$, and S there. Thus $\mathcal{H} - (Q \cap Z)$ belongs to $\mathfrak{H}^+$ and, since $\mathfrak{H}$ is hereditary, this yields the contradiction $(\mathcal{H} - Z) \in \mathfrak{H}^+$. $\square$

Proof of Theorem 1.7. If $\mathcal{H}$ satisfies $\delta(\mathcal{H}) > (\pi(\mathcal{F})/(r-1)! - \varepsilon)\,n^{r-1}$, then

$$Z_\varepsilon(\mathcal{H}) = \varnothing \quad \text{and} \quad |\mathcal{H}| = \frac{1}{r} \sum_{v \in V(\mathcal{H})} d_\mathcal{H}(v) > (\pi(\mathcal{F})/r! - \varepsilon)\,n^r.$$

Therefore, Theorem 3.2 implies Theorem 1.7. $\square$

Finally we prove Theorem 1.8 using Lemma 3.1.

Proof of Theorem 1.8. Pick $\delta \in (0, 1/2)$ and $N_0' \in \mathbb{N}$ such that $\mathcal{F}$ is $(2\delta, N_0')$-vertex-extendable with respect to $\mathfrak{H}$. The vertex stability of $\mathcal{F}$ with respect to $\mathfrak{H}$ applied to δ yields some $\varepsilon > 0$ and $N_0 \in \mathbb{N}$ (see Definition 1.2 (b)). We may assume that $\varepsilon \leqslant \delta$ and $N_0 \geqslant 2N_0'$.

Now let $\mathcal{H}$ be an $\mathcal{F}$-free r-graph on $n \geqslant N_0$ vertices with $\delta(\mathcal{H}) \geqslant \big(\pi(\mathcal{F})/(r-1)! - \varepsilon\big)n^{r-1}$. We are to prove $\mathcal{H} \in \mathfrak{H}^+$. It follows from $r|\mathcal{H}| = \sum_{v \in V(\mathcal{H})} d_\mathcal{H}(v)$ that $|\mathcal{H}| \geqslant (\pi(\mathcal{F})/r! - \varepsilon)\,n^r$. So by the vertex-stability of $\mathcal{F}$ there exists a set $B \subseteq V(\mathcal{H})$ of size at most δn such that the r-graph $\mathcal{H}' = \mathcal{H} - B$ is a member of $\mathfrak{H}^+$. Since $\delta(\mathcal{H}) \geqslant (\pi(\mathcal{F})/(r-1)! - \delta)\,n^{r-1}$ and $n \geqslant 2N_0'$ it follows from Lemma 3.1 and our choice of δ that $\mathcal{H} \in \mathfrak{H}^+$. $\square$

§4. Applications

4.1. **Graphs.** In this subsection we prove Theorem 1.9. As we have already mentioned, its part (b) is due to Erdős and Simonovits [9], who proved that for every edge-critical graph F with $\chi(F) = \ell + 1 \geqslant 3$ there exists some $N_0 \in \mathbb{N}$ such that every graph G on $n \geqslant N_0$ vertices whose minimum degree is larger than $\frac{3\ell-4}{3\ell-1}n + O(1)$ either contains F or is ℓ-colorable. Consequently, all edge-critical families of graphs are degree-stable with respect to $\mathfrak{K}_\ell$.

Now suppose, conversely, that some graph family $\mathcal{F}$ is degree-stable with respect to the class $\mathfrak{K}_\ell$, where $\ell \geqslant 2$. This means, in particular, that for every $n \geqslant \ell + 1$ the graph $T^+(n, \ell)$ obtained from the n-vertex ℓ-partite Turán graph by inserting an additional edge into one of its vertex classes cannot be $\mathcal{F}$-free. Moreover, there cannot exist a graph $F' \in \mathcal{F}$ with $\chi(F') \leqslant \ell$, for then some member of $\mathfrak{K}_\ell$ would fail to be $\mathcal{F}$-free (as demanded by Definition 1.2). So altogether, $\mathcal{F}$ needs to contain an edge-critical graph F with $\chi(F) = \chi(\mathcal{F}) = \ell + 1$. In other words, $\mathcal{F}$ is indeed edge-critical.

We are left with proving part (a) of Theorem 1.9. The forward implication from vertex stability to matching-criticality is very similar to the argument in the previous paragraph, but instead of the graphs $T^+(n, \ell)$ one considers graphs obtained from Turán graphs by

inserting (almost) perfect matchings into one of their partition classes. Omitting further details we proceed to the backwards implication. It clearly suffices to treat families consisting of a single graph.

Lemma 4.1. *Let F be a graph with $\chi(F) = \ell + 1 \geqslant 3$. If F is matching-critical, then F is vertex-stable with respect to $\mathfrak{K}_\ell$.*

Proof of Lemma 4.1. Given $\delta > 0$ we choose $\varepsilon, \eta > 0$ and $N_0 \in \mathbb{N}$ obeying the hierarchy $N_0^{-1} \ll \varepsilon \ll \eta \ll \delta$. Suppose that G is an F-free graph on $n \geqslant N_0$ vertices with at least $\left(\frac{\ell-1}{2\ell} - \varepsilon\right) n^2$ edges. We are to prove that G can be made ℓ-partite by deleting at most δn vertices. Theorem 1.1 yields a partition $V(G) = \bigcup_{i \in [\ell]} V_i$ such that $\sum_{i \in [\ell]} |G[V_i]| \leqslant \eta n^2$. Set

$$X_i = \left\{ x \in V_i \colon \text{ there is } j \in [\ell] \smallsetminus \{i\} \text{ such that } |V_j \smallsetminus N(x)| \geqslant \frac{n}{3\ell v(F)} \right\}$$

for every $i \in [\ell]$. Since $|G[V_i, V_j]| \geqslant |V_i||V_j| - 2\eta n^2$ holds for all distinct $i, j \in [\ell]$, we have $|X_i| \leqslant 6(\ell-1)\ell v(F)\eta n \leqslant \delta n/2\ell$ for every $i \in [\ell]$.

Recall that there is a matching M such that $\chi(F - M) \leqslant \ell$. If for some $i \in [\ell]$ there are $|M|$ independent edges $e_1, \ldots, e_{|M|}$ in $G[V_i \smallsetminus X_i]$ we can find a copy of F in G where these edges $e_1, \ldots, e_{|M|}$ play the rôle of M. So by $F \nsubseteq G$ such matchings do not exist and it follows that for every $i \in [\ell]$ there is a set $Y_i \subseteq V_i \smallsetminus X_i$ of size $|Y| \leqslant 2|M|$ covering all edges. Now the set $Q = \bigcup_{i \in [\ell]}(X_i \cup Y_i)$ has size at most $\delta n/2 + 2\ell|M| \leqslant \delta n$ and $G - Q$ is ℓ-partite. $\qquad\square$

4.2. **Cancellative hypergraphs and generalized triangles.** The goal of this subsection is to deduce Theorem 1.10 from Theorem 1.7. We commence by introducing a class $\mathfrak{T}_r$ of Σ_r-free r-graphs which is larger than $\mathfrak{K}_r^r$.

For integers $n \geqslant r \geqslant \ell \geqslant 1$ we call an r-graph $\mathcal{G}$ on n vertices an (n, r, ℓ)-*system* if every ℓ-subset of $V(\mathcal{G})$ is contained in at most one edge. As shown in [11, 21, 22, 33, 35], the Turán problem for Σ_r is closely related to $(n, r, r-1)$-systems. Given any $r \geqslant 3$ we write $\mathfrak{T}_r$ for the class of all blowups of 2-covered $(n, r, r-1)$-systems. Since K_r^r is a 2-covered $(r, r, r-1)$-system, we have $\mathfrak{K}_r^r \subseteq \mathfrak{T}_r$. Perhaps at first sight surprisingly, we shall apply Theorem 1.7 to $\mathcal{F} = \Sigma_r$ and $\mathfrak{H} = \mathfrak{T}_r$. This choice of $\mathfrak{H}$ is forced upon us due to the symmetrized stability assumption and the following fact.

Lemma 4.2. *For $r \geqslant 3$ a Σ_r-free r-graph is symmetrised if and only if it is a proper blowup of some 2-covered $(n, r, r-1)$-system.*

Proof of Lemma 4.2. Suppose first that $\mathcal{H}$ is a symmetrised Σ_r-free r-graph. Being a symmetrised hypergraph, $\mathcal{H}$ is a proper blowup of some 2-covered r-graph $\mathcal{T}$. If $\mathcal{T}$ fails to be a $(v(\mathcal{T}), r, r-1)$-system, then there are edges $B, C \in \mathcal{T}$ such that $|B \cap C| = r - 1$.

Since $\mathcal{T}$ is 2-covered, some edge $A \in \mathcal{T}$ contains the two-element set $B \triangle C$. Now $\{A, B, C\}$ is a subgraph of $\mathcal{T}$ belonging to Σ_r, contrary to $\mathcal{H}$ being Σ_r-free. This proves that $\mathcal{T}$ is indeed a $(v(\mathcal{T}), r, r-1)$-system.

In the converse direction, proper blowups of 2-covered $(n, r, r-1)$-systems are clearly symmetrised and an argument similar to the previous paragraph shows that they are Σ_r-free as well. $\square$

Proceeding with our intended application of Theorem 1.7 we observe that due to being closed under the formation of homomorphic images Σ_r is blow-up invariant. Moreover, $\mathfrak{T}_r$ is clearly hereditary and and the previous lemma shows that Σ_r is symmetrized-stable with respect to $\mathfrak{T}_r$. So it remains to verify vertex-extendibility for $r \in \{3, 4\}$. As the following lemma demonstrates, for this task we may restrict our attention to $\mathfrak{K}_r^r$ rather than $\mathfrak{T}_r$.

Lemma 4.3. *For $r \in \{3, 4\}$ there exists $\varepsilon_r > 0$ such that every $\mathcal{H} \in \mathfrak{T}_r$ with minimum degree $\delta(\mathcal{H}) > (r^{1-r} - \varepsilon_r)n^{r-1}$ belongs to $\mathfrak{K}_r^r$.*

Proof of Lemma 4.3. Choose $\varepsilon_3, \varepsilon_4 > 0$ sufficiently small and suppose that for some $r \in \{3, 4\}$ an r-graph $\mathcal{H} \in \mathfrak{T}_r$ has n vertices and minimum degree at least $(r^{1-r} - \varepsilon_r) n^{r-1}$. Without loss of generality we can suppose that $\mathcal{H}$ is a proper blowup of some (not necessarily 2-covered) $(m, r, r-1)$-system $\mathcal{T}$ with $V(\mathcal{T}) = [m]$. Write $\mathcal{H} = \mathcal{T}[V_1, \ldots, V_m]$ and set $x_i = |V_i|/n$ for every $i \in [m]$.

Since $d_{\mathcal{H}}(v) = L_{L_{\mathcal{T}}(i)}(x_1, \ldots, x_m)n^2$ holds for all $v \in V_i$ and $i \in [m]$, the minimum degree assumption yields $L_{L_{\mathcal{T}}(i)}(x_1, \ldots, x_m) \geq r^{1-r} - \varepsilon_r$ for every $i \in [m]$. On the other hand, as every $(r-1)$-subset of $V(\mathcal{T})$ in contained in at most one edge of $\mathcal{T}$, we have $\sum_{i \in [m]} L_{L_{\mathcal{T}}(i)}(x_1, \ldots, x_m) \leq L_{K_m^{r-1}}(x_1, \ldots, x_m)$. It follows that

$$\left(r^{1-r} - \varepsilon_r\right) m \leq \sum_{i \in [m]} L_{L_{\mathcal{T}}(i)}(x_1, \ldots, x_m) \leq \lambda(K_m^{r-1}) = \binom{m}{r-1}/m^{r-1}. \qquad (4.1)$$

Now for $r = 3$ a sufficiently small choice of ε_3 guarantees $m \in \{2, 3\}$; so $\mathcal{T}$ consists of a single edge and $\mathcal{H} \in \mathfrak{K}_3^3$. In the 4-uniform case (4.1) leads to $m \in \{4, 5\}$; since there exists no 2-covered $(5, 4, 3)$-system, the case $m = 5$ is impossible and thus we have indeed $\mathcal{H} \in \mathfrak{K}_4^4$. $\square$

Due to the lower bound $\pi(\Sigma_r) \geq r!/r^r$, which follows from the fact that r-graphs in $\mathfrak{K}_r^r$ are Σ_r-free, the next lemma will imply that for $r \in \{3, 4\}$ the family Σ_r is vertex-extendable with respect to $\mathfrak{T}_r$.

Lemma 4.4. *For every integer $r \geq 2$ there exist $\zeta > 0$ and $N_0 \in \mathbb{N}$ such that every Σ_r-free r-graph $\mathcal{H}$ on $n \geq N_0$ vertices which has minimum degree $\delta(\mathcal{H}) \geq (r^{1-r} - \zeta)n^{r-1}$ and possesses a vertex v such $\mathcal{H} - v$ is K_r^r-colorable is K_r^r-colorable itself.*

Proof of Lemma 4.4. Given $r \geqslant 2$ we choose appropriate constants $\zeta > 0$ and $N_0 \in \mathbb{N}$ fitting into the hierarchy $N_0^{-1} \ll \zeta \ll r^{-1}$. Now let $\mathcal{H}$ be a Σ_r-free r-graph on $n \geqslant N_0$ vertices whose minimum degree is at least $(r^{1-r} - \zeta)n^{r-1}$. Set $V = V(\mathcal{H})$ and suppose that some vertex $v \in V$ has the property that $\mathcal{H}_v = \mathcal{H} - v$ is K_r^r-colorable. Fix a K_r^r-coloring $V(\mathcal{H}_v) = \bigcup_{i \in [r]} V_i$ of $\mathcal{H}_v$. Clearly

$$\delta(\mathcal{H}_v) \geqslant \left(r^{1-r} - \zeta\right) n^{r-1} - n^{r-2} \geqslant \left(r^{1-r} - 2\zeta\right) n^{r-1} \tag{4.2}$$

and Corollary 2.3 (a) yields

$$|V_i| = \left(1/r \pm \zeta^{1/3}\right) n \quad \text{for all } i \in [r]. \tag{4.3}$$

Claim 4.5. *Every edge of $\mathcal{H}$ intersects every vertex class V_i in at most one vertex.*

Proof of Claim 4.5. By symmetry it suffices to show $|E \cap V_1| \leqslant 1$ for every $E \in \mathcal{H}$. Assume for the sake of contradiction that there exist distinct vertices $w_1, w_1' \in E \cap V_1$. The $(r-1)$-graphs $G_1 = L_{\mathcal{H}_v}(w_1)$ and $G_1' = L_{\mathcal{H}_v}(w_1')$ are $(r-1)$-partite with vertex partition $V_2 \cup \cdots \cup V_r$ and by (4.2) both of them have at least the size $(1/r^{r-1} - 2\zeta)n^{r-1}$. Due to (4.3) this implies $|G_1 \cap G_1'| \geqslant n^{r-1}/2r^{r-1}$ and, in particular, there exists an edge $e \in G_1 \cap G_1'$. Now $\{E, e \cup \{w_1\}, e \cup \{w_1'\}\} \in \Sigma_r$ contradicts the assumption that $\mathcal{H}$ is Σ_r-free. $\square$

Since no edge of $L_{\mathcal{H}}(v)$ can intersect all the partition classes $V_1, \ldots, V_r$ we may assume without loss of generality that at least $d(v)/r$ edges of $L_{\mathcal{H}}(v)$ are contained in $V_2 \cup \ldots \cup V_r$.

Claim 4.6. *We have $N_{\mathcal{H}}(v) \cap V_1 = \varnothing$.*

Proof of Claim 4.6. Suppose to the contrary that there exists a vertex $u \in N_{\mathcal{H}}(v) \cap V_1$ and consider an edge $E \in \mathcal{H}$ containing $\{u, v\}$. Let G_u and G_v be the subgraphs of $L_{\mathcal{H}}(u)$ and $L_{\mathcal{H}}(v)$ induced by $\bigcup_{j \in [2,r]} V_j$ respectively. Clearly, G_u is $(r-1)$-partite and by Claim 4.5 G_v is $(r-1)$-partite as well. Moreover, (4.2) yields $|G_u| \geqslant (1/r^{r-1} - 2\zeta) n^{r-1}$. Together with $|G_v| \geqslant d(v)/r \geqslant (1/r^{r-1} - \zeta) n^{r-1}/r$ and (4.3) this implies

$$|G_u \cap G_v| \geqslant \frac{1}{2r} \frac{n^{r-1}}{r^{r-1}}.$$

But if $e \in G_u \cap G_v$ is arbitrary, then the subgraph $\{E, e \cup \{v\}, e \cup \{u\}\}$ of $\mathcal{H}$ belongs to Σ_r, contrary to $\mathcal{H}$ being Σ_r-free. $\square$

By Claim 4.5 and Claim 4.6 the partition $V(\mathcal{H}) = \bigcup_{i \in [r]} \widehat{V}_i$ defined by

$$\widehat{V}_i = \begin{cases} V_1 \cup \{v\} & \text{if } i = 1, \\ V_i & \text{if } i \in [2, r], \end{cases}$$

is a K_r^r-coloring of $\mathcal{H}$. This completes the proof of Lemma 4.4. $\square$

We have thereby checked all assumptions of Theorem 1.7 and can conclude that for $r \in \{3, 4\}$ the family Σ_r is degree-stable with respect to $\mathfrak{T}_r$. In view of Lemma 4.3 this implies that Σ_r is degree-stable with respect to $\mathfrak{K}_r^r$ as well.

4.3. **Hypergraph expansions.** Throughout this subsection we fix two integers $\ell \geqslant r \geqslant 2$ and an r-graph F with $\ell + 1$ vertices satisfying the assumptions of Theorem 1.12. Our goal is to conclude from Theorem 1.7 that the family $\mathcal{K}_{\ell+1}^F$ is indeed degree-stable with respect to $\mathfrak{K}_\ell^r$.

Since the family $\mathcal{K}_{\ell+1}^F$ is closed under taking homomorphic images, it is blowup-invariant and, clearly, $\mathfrak{K}_\ell^r$ is hereditary. So it remains to show that $\mathcal{K}_{\ell+1}^F$ is symmetrized-stable and vertex-extendable with respect to $\mathfrak{K}_\ell^r$. The fact that all members of $\mathfrak{K}_\ell^r$ are $\mathcal{K}_{\ell+1}^F$-free implies $\pi(\mathcal{K}_{\ell+1}^F) \geqslant (\ell)_r/\ell^r$ and thus our claim on symmetrized stability follows from the next statement.

Lemma 4.7. *There exists some $\varepsilon > 0$ such that every symmetrized $\mathcal{K}_{\ell+1}^F$-free r-graph $\mathcal{H}$ with n vertices and $|\mathcal{H}| > \left(\binom{\ell}{r}/\ell^r - \varepsilon\right) n^r$ is K_ℓ^r-colorable.*

Proof. We contend that every positive number ε satisfing

$$\sup\left\{\lambda(\mathcal{G}) \colon \mathcal{G} \text{ is } F\text{-free but not } K_\ell^r\text{-colorable}\right\} + \varepsilon \leqslant \binom{\ell}{r}/\ell^r$$

has the desired property. To see this we consider an arbitrary symmetrized $\mathcal{K}_{\ell+1}^F$-free r-graph $\mathcal{H}$ with n vertices and $|\mathcal{H}| > \left(\binom{\ell}{r}/\ell^r - \varepsilon\right) n^r$. Since $\mathcal{H}$ is symmetrized, there exists a 2-covered hypergraph $\mathcal{G}$ such that $\mathcal{H}$ is a proper blow-up of $\mathcal{G}$. Now $|\mathcal{H}| \leqslant \lambda(\mathcal{G}) n^r$ yields $\binom{\ell}{r}/\ell^r - \varepsilon < \lambda(\mathcal{G})$. On the other hand, since $\mathcal{H}$ is $\mathcal{K}_{\ell+1}^F$-free and $\mathcal{G}$ is 2-covered, $\mathcal{G}$ must be F-free. So our choice of ε implies that $\mathcal{G}$ is K_ℓ^r-colorable and, hence, so is $\mathcal{H}$. $\qquad\square$

The next lemma implies that $\mathcal{K}_{\ell+1}^F$ is vertex-extendable with respect to $\mathfrak{K}_\ell^r$ and thus concludes the proof of Theorem 1.12.

Lemma 4.8. *There exist $\zeta > 0$ and $N_0 \in \mathbb{N}$ such that every $\mathcal{K}_{\ell+1}^F$-free r-graph $\mathcal{H}$ on $n \geqslant N_0$ vertices satisfying the minimum degree condition $\delta(\mathcal{H}) > \left(\binom{\ell-1}{r-1}/\ell^{r-1} - \zeta\right) n^{r-1}$ and possessing a vertex v such that $\mathcal{H} - v$ is K_ℓ^r-colorable is K_ℓ^r-colorable itself.*

A slight modification of the proof below shows that this holds for $H_{\ell+1}^F$ instead of the family $\mathcal{K}_{\ell+1}^F$ as well.

Proof of Lemma 4.8. Choose $N_0^{-1} \ll \zeta \ll \ell^{-1}$ appropriately and let $\mathcal{H}$ be a $\mathcal{K}_{\ell+1}^F$-free r-graph on $n \geqslant N_0$ vertices whose minimum degree is at least $\left(\binom{\ell-1}{r-1}/\ell^{r-1} - \zeta\right) n^{r-1}$. Write $V = V(\mathcal{H})$ and suppose that $\mathcal{H}_v = \mathcal{H} - v$ is K_ℓ^r-colorable for some vertex $v \in V$. Consider a K_ℓ^r-coloring $\bigcup_{i \in [\ell]} V_i = V \smallsetminus \{v\}$ of $\mathcal{H}_v$ and the associated blowup $\hat{K}_\ell^r = K_\ell^r[V_1, \ldots, V_\ell]$

of K_ℓ^r. Sets in $\widehat{K}_\ell^r \smallsetminus \mathcal{H}_v$ are called *missing edges* of $\mathcal{H}_v$; furthermore, for every $u \in V$ sets in $L_{\widehat{K}_\ell^r}(u) \smallsetminus L_{\mathcal{H}_v}(u)$ are called *missing edges* of $L_{\mathcal{H}_v}(u)$.

Notice that

$$\delta(\mathcal{H}_v) \geqslant \delta(\mathcal{H}) - n^{r-2} \geqslant \left(\binom{\ell-1}{r-1} / \ell^{r-1} - 2\zeta \right) n^{r-1}. \tag{4.4}$$

Due to $|\mathcal{H}| \geqslant n\delta(\mathcal{H})/r > \left(\binom{\ell}{r}/\ell^r - \zeta \right) n^r$ we have, similarly,

$$|\mathcal{H}_v| \geqslant |\mathcal{H}| - n^{r-1} \geqslant \left(\binom{\ell}{r} / \ell^r - 2\zeta \right) n^r. \tag{4.5}$$

Consequently, the number of missing edges of $\mathcal{H}_v$ is at most $2\zeta n^r$. We proceed with a weak version of Corollary 2.3.

Claim 4.9. *The following hold.*

  (a) *We have $|V_i| = \left(1/\ell \pm \zeta^{1/3} \right) n$ for every $i \in [\ell]$.*
  (b) *If $i \in [\ell]$ and $u \in V(\mathcal{H}_v) \smallsetminus V_i$, then $|V_i \smallsetminus N_H(u)| \leqslant \zeta^{1/3} n$.*
  (c) *For every $u \in V(\mathcal{H}_v)$ the number of missing edges of $L_{\mathcal{H}_v}(u)$ is at most $\zeta^{1/3} n^{r-1}$.* $\square$

The proof of our next claim exploits the fact that F fails to be 2-covered, i.e., that there exist two distinct vertices $u, v \in V(F)$ such that $uv \notin \partial_{r-2} F$. Indeed, if F has an isolated vertex this is clear and if $F \subseteq B(r, \ell+1)$ we can take $u = 1$ as well as $v = r + 1$.

Claim 4.10. *We have $|E \cap V_i| \leqslant 1$ for all $E \in \mathcal{H}$ and $i \in [\ell]$.*

Proof of Claim 4.10. Otherwise we may assume, without loss of generality, that for some edge E there exist two distinct vertices $w_1, w_1' \in E \cap V_1$. By Claim 4.9 (a) and (b) for every $i \in [2, \ell]$ the set $V_i' = V_i \cap N_{\mathcal{H}}(w_1) \cap N_{\mathcal{H}}(w_1')$ satisfies $|V_i'| > n/2\ell$. Applying Lemma 2.4 with $S = \{w_1, w_1'\}$ and $T = [2, \ell]$ we obtain vertices $u_i \in V_i'$ for $i \in [2, \ell]$ such that the set $U = \{u_i \colon i \in [2, \ell]\}$ satisfies $\mathcal{H}[U \cup \{w_1\}] \cong \mathcal{H}[U \cup \{w_1'\}] \cong K_\ell^r$. As F is not 2-covered, it is a subgraph of $H = \mathcal{H}[U \cup \{w_1, w_1'\}]$. Thus $H \cup \{E\}$ is a weak expansion of F, contrary to $\mathcal{H}$ being $\mathcal{K}_{\ell+1}^F$-free. $\square$

Essentially it remains to be shown that $N_{\mathcal{H}}(v) \cap V_i = \varnothing$ holds for some $i \in [\ell]$. Preparing ourselves we first show the following weaker result.

Claim 4.11. *There is no index $i \in [\ell]$ such that $N_{\mathcal{H}}(v) \cap V_i \neq \varnothing$ and $|N_{\mathcal{H}}(v) \cap V_j| \geqslant 2\zeta^{1/4r} n$ for all $j \in [\ell] \smallsetminus \{i\}$.*

Proof of Claim 4.11. By symmetry it suffices to deal with the case $i = 1$. Assume for the sake of contradiction that there exists a vertex $u_1 \in N_{\mathcal{H}}(v) \cap V_1$ and moreover, that $|N_{\mathcal{H}}(v) \cap V_j| \geqslant 2\zeta^{1/4r} n$ for all $j \in [2, \ell]$. We shall show that, contrary to the hypothesis, $\mathcal{H}$ contains a weak expansion of F.

If F has an isolated vertex, we observe that due to Claim 4.9 (b) for every $j \in [2, \ell]$ the set $V_j' = V_j \cap N_{\mathcal{H}}(v) \cap N_{\mathcal{H}}(u_1)$ has at least the size $|V_j'| \geqslant 2\zeta^{1/4r}n - \zeta^{1/3}n > \zeta^{1/4r}n$. So we can apply Lemma 2.4 to $S = \{u_1\}$ and $T = [2, \ell]$, thus obtaining a set $U = \{u_j \colon j \in [2, \ell]\}$ with $u_j \in V_j'$ for $j \in [2, \ell]$ and $\mathcal{H}[U \cup \{u_1\}] \cong K_\ell^r$. For every $i \in [\ell]$ let $e_i \in \mathcal{H}$ be an edge containing both u_i and v. Since at least one vertex of F is isolated, $H = \mathcal{H}[U \cup \{u_1\}] \cup \{e_j \colon j \in [\ell]\}$ is the desired weak $(\ell + 1)$-expansion of F.

So it remains to consider the case $F \subseteq B(r, \ell + 1)$. Pick an edge $E \in \mathcal{H}$ containing $\{v, u_1\}$. By Claim 4.10 we may assume that E is of the form $\{v, u_1, \ldots, u_{r-1}\}$, where $u_j \in V_j$ holds for all $j \in [2, r-1]$. Claim 4.9 (b) tells us that for every $k \in [r, \ell]$ the set

$$V_k' = V_k \cap N_{\mathcal{H}}(v) \cap \left(\bigcap_{j \in [1, r-1]} N_{\mathcal{H}}(u_j) \right)$$

has at least the size $|V_k'| \geqslant 2\zeta^{1/4r}n - (r-1)\zeta^{1/3}n > \zeta^{1/4r}n$. For this reason Lemma 2.4 applied to $S = \{u_1, \ldots, u_{r-1}\}$ and $T = [r, \ell]$ leads to a set $U = \{u_k \colon k \in [r, \ell]\}$ such that

- $u_k \in V_k'$ for every $k \in [r, \ell]$,
- $\mathcal{H}[U] \cong K_{\ell-r+1}^r$,
- and $L_{\mathcal{H}}(u_j)[U] \cong K_{\ell-r+1}^{r-1}$ for every $j \in [r-1]$.

Next we select for every $k \in [r, \ell]$ an edge $E_i \in \mathcal{H}$ containing both u_k and v. Now

$$H = \mathcal{H}[U \cup \{u_1, \ldots, u_{r-1}\}] \cup \{E\} \cup \{E_k \colon k \in [r, \ell]\}$$

is a weak expansion of $B(r, \ell + 1)$ and, a fortiori, a weak expansion of F. □

Let us now consider the set

$$S = \left\{ i \in [\ell] \colon |N_{\mathcal{H}}(v) \cap V_i| \geqslant 2\zeta^{1/4r}n \right\}.$$

By Claim 4.11 we know, in particular, that $S \neq [\ell]$. Pick an arbitrary $i_\star \in [\ell] \setminus S$. Now Claim 4.9 (a) and $|d_{\mathcal{H}}(v)| \geqslant \left(\binom{\ell-1}{r-1} / \ell^{r-1} - \zeta \right) n^{r-1}$ imply $S = [\ell] \setminus \{i_\star\}$ and a further application of Claim 4.11 discloses $N_{\mathcal{H}}(v) \cap V_{i_\star} = \varnothing$. Together with Claim 4.10 this shows that the partition $V(\mathcal{H}) = \bigcup_{i \in [\ell]} \widehat{V_i}$ defined by

$$\widehat{V_i} = \begin{cases} V_{i_\star} \cup \{v\} & \text{if } i = i_\star, \\ V_i & \text{if } i \neq i_\star, \end{cases}$$

is a K_ℓ^r-coloring of $\mathcal{H}$. This completes the proof of Lemma 4.8. □

4.4. Expansions of Matchings of size 2.

In this subsection we shall derive Theorem 1.14 from Theorem 1.7. Again it is easy to see that $\mathcal{K}_{2r}^{M_2^r}$ is blowup-invariant and that the class $\mathfrak{S}^r$ is hereditary. Bene Watts, Norin, and Yepremyan proved in [2] that

$$\sup \left\{ \lambda(\mathcal{G}) \colon \mathcal{G} \text{ is } M_2^r\text{-free but not semibipartite} \right\} < \frac{(1 - 1/r)^{r-1}}{r!}$$

holds for all $r \geqslant 4$, where, let us recall, the numerator is the supremum of the edge densities of semibipartite r-graphs. Following the proof of Lemma 4.7 one easily deduces from this result that $\mathcal{K}_{2r}^{M_2^r}$ is symmetrized-stable with respect to $\mathfrak{S}^r$. So it only remains to establish vertex-extendibility, i.e., the following lemma.

Lemma 4.12. *Let* $r \geqslant 4$ *and* $F = M_2^r$. *There exist* $\zeta > 0$ *and* $N_0 \in \mathbb{N}$ *such that every* $\mathcal{K}_{2r}^F$-*free* r-*graph* $\mathcal{H}$ *on* $n \geqslant N_0$ *vertices satisfying* $\delta(\mathcal{H}) \geqslant \left(\left(1 - \frac{1}{r}\right)^{r-1} /(r-1)! - \zeta \right) n^{r-1}$ *and possessing a vertex* v *for which* $\mathcal{H} - v$ *is semibipartite is semibipartite itself.*

In order to estimate the sizes of the vertex classes of semibipartite hypergraphs with almost the maximum number of edges we use the following estimate.

Fact 4.13. *If* $r \geqslant 2$ *and* $x \in [0, 1]$, *then*

$$\frac{x(1-x)^{r-1}}{(r-1)!} + \frac{1}{r!}\left(1 - \frac{1}{r}\right)^{r-3}\left(x - \frac{1}{r}\right)^2 \leqslant \frac{1}{r!}\left(1 - \frac{1}{r}\right)^{r-1}.$$

Note that equality holds for $x = 1/r$ and $x = 1$.

Proof of Fact 4.13. The case $x = 1$ being clear we assume $x \in [0, 1)$ from now on. The standard inductive proof of Bernoulli's inequality also shows $(1 + 2h)(1 + h)^{r-2} \geqslant 1 + rh$ for every real $h \geqslant -1$. In particular, for $h = (x - 1/r)/(1 - x)$ we obtain

$$\left(\frac{1 - 1/r}{1 - x}\right)^{r-2}\frac{1 + x - 2/r}{1 - x} \geqslant \frac{(r-1)x}{1 - x}.$$

Multiplying by $(1 - x)^r$ we deduce

$$(r-1)x(1-x)^{r-1} \leqslant (1 - 1/r)^{r-2}(1 - x)(1 + x - 2/r)$$
$$= (1 - 1/r)^{r-2}[(1 - 1/r)^2 - (x - 1/r)^2]$$

and now it remains to divide by $(r-1)(r-1)!$. $\qquad\square$

Proof of Lemma 4.12. Fix some sufficiently small $\zeta \ll r^{-1}$ and then some sufficiently large $N_0 \gg \zeta^{-1}$. Let $\mathcal{H}$ be a $\mathcal{K}^F$-free r-graph on $n \geqslant N_0$ vertices whose minimum degree is at least $\left(\left(1 - \frac{1}{r}\right)^{r-1} /(r-1)! - \zeta \right) n^{r-1}$. Set $V = V(\mathcal{H})$ and suppose that for some vertex $v \in V$ the r-graph $\mathcal{H}_v = \mathcal{H} - v$ is semibipartite. Fix a partition $V(\mathcal{H}_v) = V_1 \cup V_2$ such that $|E \cap V_1| = 1$ holds for every $E \in \mathcal{H}_v$ and let $\hat{\mathcal{S}}$ be the complete semibipartite r-graph on $V(\mathcal{H}_v)$ corresponding to this partition. Sets in $\hat{\mathcal{S}} \smallsetminus \mathcal{H}_v$ are called *missing edges of* $\mathcal{H}_v$, and for every $u \in V \smallsetminus \{v\}$ sets in $L_{\hat{\mathcal{S}}}(u) \smallsetminus L_{\mathcal{H}_v}(u)$ are called *missing edges of* $L_{\mathcal{H}_v}(u)$.

As usual we have

$$\delta(\mathcal{H}_v) \geqslant \left(\left(1 - \frac{1}{r}\right)^{r-1} /(r-1)! - 2\zeta \right) n^{r-1} \quad \text{and} \quad |\mathcal{H}_v| \geqslant \left(\left(1 - \frac{1}{r}\right)^{r-1} /r! - 2\zeta \right) n^r.$$

In particular, the number of missing edges of $\mathcal{H}_v$ is at most $2\zeta n^r$.

Claim 4.14. *The following statements hold.*

 (*a*) *We have* $|V_1| = \left(1/r \pm \zeta^{1/3}\right) n$ *and* $|V_2| = \left((r-1)/r \pm \zeta^{1/3}\right) n$.

 (*b*) *For every* $u \in V(\mathcal{H}_v)$ *the number of missing edges of* $L_{\mathcal{H}_v}(u)$ *is at most* $\zeta^{1/3} n^{r-1}$.

 (*c*) *If* $u \in V_1$, *then* $|V_2 \setminus N_{\mathcal{H}_v}(u)| \leqslant \zeta^{1/3} n$.

 (*d*) *If* $u \in V_2$, *then* $|N_{\mathcal{H}_v}(u)| \geqslant \left(1 - \zeta^{1/3}\right) n$.

Proof. Setting $x = |V_1|/n$ we have

$$2\zeta > \frac{(1 - 1/r)^{r-1}}{r!} - \frac{|\widehat{\mathcal{S}}|}{n^r} > \frac{(1 - 1/r)^{r-1}}{r!} - \frac{x(1-x)^{r-1}}{(r-1)!}$$

and due to $\zeta \ll r^{-1}$ Fact 4.13 leads to $|x - 1/r| \leqslant O_r(\zeta^{1/2}) \leqslant \zeta^{1/3}$, which proves (*a*). Moroever, in $\widehat{\mathcal{S}}$ every vertex has degree $\left(\left(1 - \frac{1}{r}\right)^{r-1}/(r-1)! \pm O_r(\zeta^{1/2})\right) n^{r-1}$ and thus for every $u \in V(\mathcal{H}_v)$ there are at most $O_r(\zeta^{1/2}) n^{r-1}$ missing edges of $L_{\mathcal{H}_v}(u)$, which implies (*b*). Now for part (*c*) it suffices to observe that every vertex in $|V_2 \setminus N_{\mathcal{H}_v}(u)|$ belongs to $\Omega_r(n^{r-2})$ missing edges of $L_{\mathcal{H}_v}(u)$ and the argument for (*d*) is similar. □

Since $\mathcal{H}$ contains no weak expansion of M_2^r, there cannot exist two disjoint edges $E, E' \in \mathcal{H}$ such that $E \cup E'$ is 2-covered.

Claim 4.15. *If two distinct vertices* $u, w \in V(\mathcal{H})$ *satisfy*

$$|L_{\mathcal{H}}(u)[V_2]|, |L_{\mathcal{H}}(w)[V_2]| \geqslant \left(\left(1 - \frac{1}{r}\right)^{r-1}/(r-1)! - \zeta^{1/4}\right) n^{r-1},$$

then no edge of $\mathcal{H}$ *contains both of them.*

Proof of Claim 4.16. Assume contrariwise that some edge $E \in \mathcal{H}$ contains u and w. We shall show that this leads to two disjoint edges E_u, E_w of $\mathcal{H}$ such that $u \in E_u \subseteq V_2 \cup \{u\}$, $w \in E_w \subseteq V_2 \cup \{w\}$, and $E_u \cup E_w$ is 2-covered, which is absurd.

Owing to Claim 4.14 (*a*) and our assumption on the links of u and w we have

$$|V_2 \setminus N_{\mathcal{H}}(u)|, |V_2 \setminus N_{\mathcal{H}}(w)| \leqslant \zeta^{1/5} n.$$

The latter estimate and our lower bound on $|L_{\mathcal{H}}(u)[V_2]|$ show that there exists an edge $E_u \in \mathcal{H}$ such that $u \in E_u$ and $E_u \setminus \{u\} \subseteq V_2 \cap N_{\mathcal{H}}(w)$. Now Claim 4.14 (*d*) and our upper bound on $|V_2 \setminus N_{\mathcal{H}}(u)|$ imply that the set $V_2' = \bigcap_{x \in E_u} N_{\mathcal{H}}(x) \cap (V_2 \setminus E_u)$ has at least the size $|V_2'| \geqslant |V_2| - 2\zeta^{1/5} n$. Thus there exists an edge $E_w \in \mathcal{H}_v$ with $w \in E_w \subseteq V_2' \cup \{w\}$. Clearly E_u and E_w are as desired. □

By our lower bound on $\delta(\mathcal{H}_v)$ any two distinct vertices $u, w \in V_1$ satisfy the hypothesis of Claim 4.15, which has the following consequence.

Claim 4.16. *We have* $|E \cap V_1| \leqslant 1$ *for every* $E \in \mathcal{H}$. □

Notice that $d_{\mathcal{H}}(v) \geqslant \left(\left(1 - \frac{1}{r}\right)^{r-1} / (r-1)! - \zeta \right) n^{r-1}$ yields

$$|N_{\mathcal{H}}(v)| \geqslant (1 - 1/r - O_r(\zeta))\, n \geqslant 2n/3,$$

whence

$$|N_{\mathcal{H}}(v) \cap V_2| \geqslant 2n/3 - |V_1| \geqslant n/3. \tag{4.6}$$

If there exists no edge $E_\star \in \mathcal{H}$ with $v \in E_\star \subseteq V_2 \cup \{v\}$, then $V(\mathcal{H}) = V_1 \uplus (V_2 \cup \{v\})$ is a partition exemplifying that $\mathcal{H}$ is semibipartite and we are done. So we may suppose from now on that such an edge $E_\star$ exists. Consider the set $X = \bigcap_{w \in E_\star} N_{\mathcal{H}}(w)$. On the one hand, Claim 4.14 (d) and (4.6) imply

$$|X \cap V_2| \geqslant |N_{\mathcal{H}}(v) \cap V_2| - (r-1)\zeta^{1/3} n \geqslant n/3 - n/12 = n/4.$$

On the other hand, there cannot exist an edge $E' \subseteq X \smallsetminus E_\star$, for then $\{E_\star, E'\}$ would be a matching in $\mathcal{H}$ such that $E_\star \cup E'$ is 2-covered. Since there are at most $2\zeta n^r$ missing edges, this implies $|X \cap V_1| \leqslant O_r(\zeta)n \leqslant \zeta^{1/3} n$. As Claim 4.14 (d) yields $|N_{\mathcal{H}}(v) \smallsetminus X| \leqslant (r-1)\zeta^{1/3} n$, we may conclude

$$|N_{\mathcal{H}}(v) \cap V_1| \leqslant |N_{\mathcal{H}}(v) \smallsetminus X| + |X \cap V_1| \leqslant r\zeta^{1/3} n,$$

whence

$$|L_{\mathcal{H}}(v)[V_2]| \geqslant d_{\mathcal{H}}(v) - |N_{\mathcal{H}}(v) \cap V_1| n^{r-2} \geqslant \left((1 - 1/r)^{r-1} / (r-1)! - \zeta^{1/4} \right) n^{r-1}.$$

Now Claim 4.15 discloses $N_{\mathcal{H}}(v) \cap V_1 = \varnothing$. In view of Claim 4.16 this shows that the partition $V(\mathcal{H}) = (V_1 \cup \{v\}) \cup V_2$ witnesses the semibipartiteness of $\mathcal{H}$. $\qquad\square$

§5. Concluding remarks

• In this article we provided a framework for proving the degree-stability of certain classes of graph and hypergraph families, and applied it to the degree-stability of Σ_3, Σ_4, and $\mathcal{K}_\ell^F$ for some combinations of F and ℓ. In fact, one could push our results further and show that T_3, T_4, and H_ℓ^F (for some combinations of F and ℓ) are degree-stable by using the degree-stability results obtained here, applying the Removal lemma to prove the vertex-stability of T_3, T_4, and H_ℓ^F, respectively, and finally applying Theorem 1.8.

• Generalizing Theorem 1.9 one may attempt to characterize for arbitrary $\ell \geqslant r \geqslant 2$ the hypergraph families which are vertex-stable or degree-stable with respect to $\mathfrak{K}_\ell^r$. This problem is presumably very difficult and even partial results in this direction would be interesting.

• A classical example in hypergraph Turán theory suggested by Vera T. Sós is the Fano plane, i.e. the 3-graph on vertex set [7] with edge set

$$\{123, 345, 561, 174, 275, 376, 246\}.$$

The Turán density of the Fano plane was determined by De Caen and Füredi in [5]. Later Keevash and Sudakov [19] and, independently, Füredi and Simonovits [13] proved the degree-stability of the Fano plane and used it to determine the Turán number for large n. The complete determination of its Turán number was obtained only recently by Bellmann and the third author [1]. We do not know whether our method can be used to give another proof of the degree-stability of the Fano plane.

• Recall that by Theorem 1.1 every family $\mathcal{F}$ of graphs with $\chi(\mathcal{F}) = \ell + 1$ is edge-stable with respect to the family $\{T(n, \ell) \colon n \in \mathbb{N}\}$ of Turán graphs, which has the property that for every $n \in \mathbb{N}$ it contains a *unique n-vertex graph*. This state of affairs prompted the second author [29] to define for every nondegenerate family $\mathcal{F}$ of r-graphs the (edge-) stability number $\xi_e(\mathcal{F})$ to be the least number t such that there exists a class of r-graphs $\mathfrak{H}$ with the following properties:

  • $\mathcal{F}$ is edge-stable with respect to $\mathfrak{H}$;
  • for every $n \in \mathbb{N}$ there are t hypergraphs on n vertices in $\mathfrak{H}$.

For instance, the families studied in this article have stability number 1 and standard conjectures imply that the stability number of K_4^3 is infinite. It was shown recently [23, 24] that for every $t \in \mathbb{N}$ there exists a family $\mathcal{M}_t$ of triple systems such that $\xi_e(\mathcal{M}_t) = t$.

In analogy with Definition 1.2 one can also define a *vertex-stability number* $\xi_v(\mathcal{F})$ and a *degree-stability number* $\xi_d(\mathcal{F})$. These satisfy the easy estimates $\xi_e(\mathcal{F}) \leqslant \xi_v(\mathcal{F}) \leqslant \xi_d(\mathcal{F})$ and it would be interesting to study how "exotic" these parameters can get.

• Our method can also be used in the context of other combinatorial structures, such as families of edge-weigthed graphs. To give an example, we recall the following result of Erdős, Hajnal, Sós, and Szemerédi [7] from *Ramsey-Turán theory*: For $r \geqslant 2$ every K_{2r}-free graph with n vertices and more than $\left(\frac{3r-5}{3r-2} + o(1)\right)n^2/2$ edges contains an independent set of size $o(n)$. Here the constant $\frac{3r-5}{3r-2}$ is optimal and the analogous problem with forbidden cliques of odd order is much easier. The proof of this result involves a certain family $\mathcal{F}_{2r}$ of graphs with weights from $\{0, 1/2, 1\}$ assigned to their edges. The main points of the argument are (*i*) that $\pi(\mathcal{F}_{2r}) = \frac{3r-5}{3r-2}$ and (*ii*) that the regularity method establishes a connection between $\mathcal{F}_{2r}$ and K_{2r}. Lüders and the third author [25] recently obtained the sharper result that for $\delta \ll r^{-1}$ every K_{2r}-free graph with n vertices and more than $\left(\frac{3r-5}{3r-2} + \delta - \delta^2\right)n^2/2$ edges contains an independent set of size δn, where the term $\frac{3r-5}{3r-2} + \delta - \delta^2$ is again optimal. Their proof requires some stability result for the family $\mathcal{F}_{2r}$. In fact, they provide a rather ad-hoc proof of vertex-stability (see [25, Proposition 5.5]) and returned to the topic in [26]

proving degree-stability. A straightforward adaptation of the Ψ-trick to weighted graphs yields an alternative (and shorter) proof of the degree-stability of $\mathcal{F}_{2r}$.

• We would like to emphasize that the strongest general stability result in this article, Theorem 3.2, can also be used for giving reasonable quantitative versions of edge stability. For instance, combined with the results in Subsection 4.2 it tells us that if $\varepsilon > 0$ is sufficiently small, then every Σ_4-free quadruple system $\mathcal{H}$ on a sufficiently large number n of vertices with more than $(1/256 - \varepsilon)n^4$ edges admits a partition $V(\mathcal{H}) = A \cup B \cup C \cup D \cup Z$ such that $|Z| \leqslant \varepsilon^{1/2}n$ and $\mathcal{H} - Z$ is 4-partite with vertex classes A, B, C, and D. Moreover, all vertices in $V(\mathcal{H}) \smallsetminus Z$ have at least the degree $(1/64 - 2\varepsilon^{1/2})n^3$. Now a careful calculation shows $|A|, |B|, |C|, |D| = (1/4 \pm 6\varepsilon^{1/2})n$ and the proof of Claim 4.5 discloses that the sets A, B, C, and D are independent in $\partial_2 \mathcal{H}$. By the proof of Claim 4.6, if some $z \in Z$ satisfies $|L_{\mathcal{H}}(z)[A, B, C]| \geqslant 4\varepsilon^{1/2}n^3$, then z has no neighbours in D. So $\mathcal{H}$ can be made 4-partite by the deletion of at most $17\varepsilon n^4$ edges, namely (i) at most εn^4 edges with two or more vertices in Z; (ii) at most $4\varepsilon n^4$ edges $zabc$ with $z \in Z$, $a \in A$, $b \in B$, $c \in C$, and $|L_{\mathcal{H}}(z)[A, B, C]| \geqslant 4\varepsilon^{1/2}n^3$; (iii) and, similarly, at most $4\varepsilon n^4$ edges of each of the three the three types $zabd$, $zacd$, $zbcd$. In particular, the edge stability of Σ_4 with respect to $\mathfrak{K}_4^4$ holds with a linear dependence between the error terms. Taking into account that at most $400\varepsilon n$ vertices $v \in V(\mathcal{H})$ can satisfy $d_{\mathcal{H}}(v) \leqslant n^3/80$ one can show the stronger result that $\mathcal{H}$ can be made K_4^4-colorable by the deletion of $7000\varepsilon^{3/2}n^4$ edges, which seems to be a new result.

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