

Hypergraphs without complete partite subgraphs

Dhruv Mubayi*

November 16, 2025

Abstract

Fix integers $r \geq 2$ and $1 \leq s_1 \leq \dots \leq s_{r-1} \leq t$ and set $s = \prod_{i=1}^{r-1} s_i$. Let $K = K(s_1, \dots, s_{r-1}, t)$ denote the complete r -partite r -uniform hypergraph with parts of size $s_1, \dots, s_{r-1}, t$. We prove that the Zarankiewicz number $z(n, K) = n^{r-1/s-o(1)}$ provided $t > 3^{s+o(s)}$. Previously this was known only for $t > ((r-1)(s-1))!$ due to Pohoata and Zakharov. Our novel approach, which uses Behrend's construction of sets with no 3 term arithmetic progression, also applies for small values of s_i , for example, it gives $z(n, K(2, 2, 7)) = n^{11/4-o(1)}$ where the exponent $11/4$ is optimal, whereas previously this was only known with 7 replaced by 721.

1 Introduction

Write $K = K(s_1, \dots, s_r)$ for the complete r -partite r -uniform hypergraph (henceforth r -graph) with parts of size $s_1 \leq s_2 \leq \dots \leq s_r$. More precisely, the vertex set of K comprises disjoint sets $S_1, \dots, S_r$, where $|S_i| = s_i$ for $1 \leq i \leq r$, and the edge set of K is

$$\{\{x_i, \dots, x_r\} : (x_1, \dots, x_r) \in S_1 \times \dots \times S_r\}.$$

Given K as above, write $\text{ex}(n, K)$ for the maximum number of edges in an n -vertex r -graph that contains no copy of K as a subhypergraph. Similarly, write $z(n, K)$ for the maximum number of edges in an r -partite r -graph H with parts $X_1, \dots, X_r$, each of size n , such that

*Department of Mathematics, Statistics and Computer Science, University of Illinois, Chicago, IL 60607. Email: mubayi@uic.edu. Research partially supported by NSF Award DMS-2153576.

there is no copy of $K(s_1, \dots, s_r)$ in H with $S_i \subset X_i$ for all $1 \leq i \leq r$ (there could be copies of K in H , where for some i , $S_i \not\subset X_i$). Determining $\text{ex}(n, K) = \text{ex}_r(n, K)$ is usually called the Turán problem, while determining $z(n, K) = z_r(n, K)$ is called the Zarankiewicz problem (we will omit the subscript r if it is obvious from context). These are fundamental questions in combinatorics with applications in analysis [1, 7], number theory [16], group theory [13], geometry [10], and computer science [3].

A basic result in extremal hypergraph theory, due to Erdős [9], is the upper bound

$$\text{ex}(n, K(s_1, \dots, s_r)) = O(n^{r-1/s}), \quad (1)$$

where $s = s_1 s_2 \cdots s_{r-1}$ (and, as before $s_1 \leq s_2 \leq \cdots \leq s_{r-1} \leq s_r$). Here $s_1, \dots, s_r$ are fixed and the asymptotic notation is taken as $n \rightarrow \infty$. As r is fixed, and $z(n, K(s_1, \dots, s_r)) \leq \text{ex}(rn, K(s_1, \dots, s_r))$, the same upper bound as (1) holds for $z(n, K(s_1, \dots, s_r))$.

A major problem in extremal (hyper)graph theory is to obtain corresponding lower bounds to (1) (or prove that no such lower bounds exist). In fact, it was conjectured in [15] that the exponent $r - 1/s$ in (1) is optimal. This question has been studied for graphs since the 1930s, and results of Erdős-Rényi and Brown [5] gave optimal (in the exponent) lower bounds for $K(2, t)$ and $K(3, t)$. The first breakthrough for arbitrary s_1 occurred in the mid 1990's by Kollar-Ronyai-Szabo [11] and then Alon-Ronyai-Szabo [2], who proved that $\text{ex}(n, K(s_1, s_2)) = \Omega(n^{2-1/s_1})$ as long as $s_2 > (s_1 - 1)!$. More recently, in another significant advance, Bukh [6] has proved the same lower bound as long as $s_2 > 9^{s_1+o(s_1)}$.

For $r \geq 3$, the first nontrivial constructions that were superior to the bound given by the probabilistic deletion method were provided in the cases $s_1 = \cdots = s_{r-2} = 1$ and $K(2, 2, 3)$ by the current author [15] and, soon after for $K(2, 2, 2)$ by Katz-Krop-Maggioni [12] (see also [8] for recent results on the r -uniform case $K(2, \dots, 2)$ that are superior to the probabilistic deletion bound but not optimal in the exponent). Later, optimal bounds for both $\text{ex}(n, K)$ and $z(n, K)$ were provided by Ma, Yuan, Zhang [14] (and independently by Verstraëte) by extending the method of Bukh, however, the threshold for s_r for which the bound holds was not even explicitly calculated. More recently, lower bounds matching the exponent $r - 1/s$ from (1) have been proved for $s_r > ((r - 1)(s - 1))!$ by Pohoata and Zakharov [17]. Here we improve this lower bound on s_r substantially in the Zarankiewicz case, from factorial to exponential at the expense of a small $o(1)$ error parameter in the exponent. The following is our main result.

Theorem 1. *Fix $r \geq 3$, and positive integers $s_1, \dots, s_{r-1}, t$. Then as $n \rightarrow \infty$,*

$$z_r(n, K(s_1, \dots, s_{r-1}, t)) > n^{1-o(1)} \cdot z_{r-1}(n, K(s_1, \dots, s_{r-3}, s_{r-2}s_{r-1}, t)).$$

Applying Theorem 1 repeatedly (or doing induction on r) yields

$$z_r(n, K(s_1, \dots, s_{r-1}, t)) > n^{r-2-o(1)} \cdot z_2(n, K(s, t))$$

where $s = s_1 \cdots s_{r-1}$. Bukh [6] proved that $z(n, K(s, t)) = \Omega(n^{2-1/s})$ provided $t > 3^{s+o(s)}$ and this yields the following corollary.

Corollary 2. *Fix $r \geq 2$, and integers $1 \leq s_1 \leq \dots \leq s_{r-1} < t$ where $t > 3^{s+o(s)}$ and $s = s_1 \cdots s_{r-1}$. Then as $n \rightarrow \infty$,*

$$z_r(n, K(s_1, \dots, s_{r-1}, t)) = n^{r-1/s-o(1)}.$$

We remark that Theorem 1 can also be applied for small values of s_i . For example, using the result of Alon-Rónyai-Szabó [2] that $z(n, K(4, 7)) = \Omega(n^{7/4})$, it gives

$$z(n, K(2, 2, 7)) > n^{1-o(1)} z(n, K(4, 7)) > n^{1-o(1)} n^{7/4} = n^{11/4-o(1)},$$

where the exponent $11/4$ is tight. For contrast, the previous best result due to Pohoata and Zakharov [17] yields only $z(n, K(2, 2, 721)) > \Omega(n^{11/4})$. If, as is widely believed, $z(n, K(4, 4)) = \Omega(n^{7/4})$, then this would imply via Theorem 1, that $z(n, K(2, 2, 4)) = n^{11/4-o(1)}$.

2 Proof

Write $e(H) = |E(H)|$ for a hypergraph H . To prove Theorem 1, we need the following well-known consequence of Behrend's construction [4] of a subset of $[n]$ with no 3-term arithmetic progression (see, e.g., [18]). There exists a bipartite graph G with parts of size n and $n^{2-o(1)}$ edges whose edge set is a union of n induced matchings. More precisely, there are pairwise disjoint matchings $M_1, \dots, M_n$ such that $E(G) = \cup_{i=1}^n M_i$ and for all i, j the edge set $M_i \cup M_j$ contains no path with three edges. Additionally, $e(G) = \sum_i |M_i| = n^{2-o(1)}$.

Proof of Theorem 1. Let H' be an $(r-1)$ -partite $(r-1)$ -graph with parts $X_1, \dots, X_{r-3}, [n]$ and Y each of size n with $e(H') = z_{r-1}(n, K')$ that contains no copy of the complete $(r-1)$ -partite $(r-1)$ -graph $K' = K(s_1, \dots, s_{r-3}, s_{r-2}s_{r-1}, t)$. Here we only assume that there are no copies of K' where the i th part of size s_i is a subset of X_i for $1 \leq i \leq r-3$, the $r-2$ part of size $s_{r-2}s_{r-1}$ is a subset of $[n]$, and the r th part of size t is a subset of Y . Write $d(j)$ for the degree in H' of vertex $j \in [n]$, so $e(H') = \sum_j d(j)$. By relabeling we may assume that $d(1) \geq d(2) \geq \dots \geq d(n)$.

Let G be a bipartite graph with parts A and B , each of size n comprising n induced matchings $M_1, \dots, M_n$, with $e(G) = \sum_i |M_i| = n^{2-o(1)}$. Moreover, we may assume that $|M_1| \geq |M_2| \geq \dots \geq |M_n|$.

Now define the r -partite r -graph H as follows: the parts of H are $X_1, \dots, X_{r-3}, A, B, Y$, each of size n . For each $j \in [n]$, let

$$E_j = \{\{x_1, \dots, x_{r-3}, a, b, y\} : \{x_1, \dots, x_{r-3}, j, y\} \in E(H'), (a, b) \in A \times B, \{a, b\} \in M_j\}$$

and let $E(H) = \cup_{j=1}^n E_j$. Observe that $e(H) = \sum_j d(j)|M_j|$. In words, we have replaced vertex j that lies in edge $\{x_1, \dots, x_{r-3}, j, y\}$ of H' by all possible pairs ab of M_j to create $|M_j|$ edges of H . Now Chebyshev's sum inequality and $e(G) = n^{2-o(1)}$ yield

$$\frac{1}{n} \sum_{j=1}^n d(j)|M_j| \geq \frac{1}{n^2} \sum_{j=1}^n d(j) \sum_{j=1}^n |M_j| = \frac{1}{n^2} e(H')e(G) = e(H')n^{-o(1)}.$$

Hence $e(H) = \sum_j d(j)|M_j| = n^{1-o(1)}e(H')$ as required.

Now suppose there is a copy L of $K = K(s_1, \dots, s_{r-1}, t)$ in H where the part of size s_i lies in X_i for $1 \leq i \leq r-3$, the part A' of size s_{r-2} lies in A , the B' part of size s_{r-1} lies in B , and the part of size t lies in Y . Then all $s_{r-2}s_{r-1}$ pairs in $\binom{V(L)}{2}$ within $A' \times B'$ must come from different matchings M_i as the matchings are induced. Indeed, if there is an i such that ab and $a'b'$ are distinct edges of M_i , where $a, a' \in A'$ and $b, b' \in B'$, then ab' cannot lie in any edge of H , as M' is an induced matching, but ab' must lie in many edges of L , contradiction. The number of these matchings M_i is therefore $|A'||B'| = s_{r-2}s_{r-1}$ and each such matching M_j corresponds to a vertex j of $[n]$. This means that we have a forbidden copy of K' in H' , contradiction. $\square$

Remarks.

- One shortcoming of our approach is that it applies only to the Zarankiewicz problem and not the Turán problem. It would be interesting to rectify this.
- For some r -partite r -graphs H one can define an appropriate $(r-1)$ -partite $(r-1)$ -graph H' such that $z_r(n, H) > n^{1-o(1)}z_{r-1}(n, H')$. This may give some further new results for hypergraphs.
- For the proof, we do not need that the matchings M_i are induced, we only needed that for any two edges ab and $a'b'$ of M_i , either ab' or $a'b$ is not in any other matching. But this relaxed property doesn't seem to help improve the $n^{o(1)}$ error term.

Acknowledgment. The authors is grateful to a referee for their careful reading of the paper.

References

- [1] S. V. Astashkin, K. V. Lykov, Sparse Rademacher chaos in symmetric spaces. (Russian. Russian summary) *Algebra i Analiz* 28 (2016), no. 1, 3–31; translation in *St. Petersburg Math. J.* 28 (2017), no. 1, 1–20
- [2] N. Alon, L. Rónyai, T. Szabó, Norm-graphs: variations and applications, *J. Combin. Theory Ser. B* 76 (1999), no. 2, 280–290.
- [3] L. Babai, A. Gál, J. Kollár, L. Rónyai, T. Szabó, A. Wigderson, Extremal bipartite graphs and superpolynomial lower bounds for monotone span programs, *Proceedings of the Twenty-eighth Annual ACM Symposium on the Theory of Computing* (Philadelphia, PA, 1996), 603–611.
- [4] Behrend, F. A, On sets of integers which contain no three terms in arithmetical progression, *Proc. Nat. Acad. Sci. U.S.A.* 32 (1946), 331–332.
- [5] W. G. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* 9 (1966), 281–285.
- [6] B. Bukh, Extremal graphs without exponentially small bicliques. *Duke Math. J.* 173 (2024), no. 11, 2039–2062.
- [7] A. Carbery, M. Christ, J. Wright, Multidimensional van der Corput and sublevel set estimates. *J. Amer. Math. Soc.* 12 (1999), no. 4, 981–1015.
- [8] D. Conlon, C. Pohoata, D. Zakharov, Random multilinear maps and the Erdős box problem. *Discrete Anal.* 2021, Paper No. 17, 8 pp.
- [9] P. Erdős, On extremal problems of graphs and generalized graphs. *Israel J. Math.* 2 (1964), 183–190.
- [10] J. Fox, Jacob, J. Pach, A. Suk, A polynomial regularity lemma for semialgebraic hypergraphs and its applications in geometry and property testing. *SIAM J. Comput.* 45 (2016), no. 6, 2199–2223.
- [11] J. Kollár, L. Rónyai, and T. Szabó, Norm-graphs and bipartite Turán numbers, *Combinatorica* 16 (1996), no. 3, 399–406.

- [12] N. H. Katz, E. Krop, M. Maggioni, Remarks on the box problem. *Math. Res. Lett.* 9 (2002), no. 4, 515–519.
- [13] A. Lucchini, Applying extremal graph theory to a question on finite groups. *Acta Math. Hungar.* 168 (2022), no. 1, 295–299.
- [14] J. Ma, X. Yuan, M. Zhang, Some extremal results on complete degenerate hypergraphs. *J. Combin. Theory Ser. A* 154 (2018), 598–609.
- [15] D. Mubayi, Some exact results and new asymptotics for hypergraph Turán numbers. *Combin. Probab. Comput.* 11 (2002), no. 3, 299–309.
- [16] A. Oppenheim, M. Shusterman, Squarefree polynomials with prescribed coefficients. *J. Number Theory* 187 (2018), 189–197.
- [17] C. Pohoata, D. Zakharov, Norm hypergraphs, *Combinatorica*, to appear.
- [18] Ruzsa, I. Z. and Szemerédi, E, Triple systems with no six points carrying three triangles, in *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976)*, Vol. II, Colloq. Math. Soc. János Bolyai, vol. 18, North-Holland, Amsterdam-New York, 1978, 939–945.