Coloring unions of nearly disjoint hypergraph cliques

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Abstract

We consider the maximum chromatic number of hypergraphs consisting of cliques that have pairwise small intersections. Designs of the appropriate parameters produce optimal constructions, but these are generally known to exist only when the number of cliques is exponential in the clique size [13, 18, 20]. We construct near designs where the number of cliques is polynomial in the clique size, and show that they have large chromatic number.

The case when the cliques have pairwise intersections of size at most one seems particularly challenging. Here we give lower bounds by analyzing a random greedy hypergraph process. We also consider the related question of determining the maximum number of caps in a finite projective/affine plane and obtain nontrivial upper and lower bounds.

1 Introduction

For $1 \le \ell < k \le q$, an ℓ-(q,k)-*system* is a k-uniform hypergraph (henceforth k-graph) whose edge set is the union of cliques with q vertices that pairwise share at most ℓ vertices. Such hypergraphs are ubiquitous in combinatorics. Here are some examples:

- ℓ-(q,k)-systems are extremal examples for many well-studied questions in extremal set theory, for example, an old open conjecture of Erdős [10] states that the maximum number of triples in an n-vertex triple system with no two disjoint pairs of edges with the same union is achieved by 1-$(5,2)$-systems.
- Recently, Liu, the first author and Reiher [19] constructed the first family of hypergraphs that fail to have the stability property and ℓ-(q,k)-systems were a crucial ingredient in constructing the extremal examples.

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- Classical old open questions in projective geometry ask for the maximum size of caps in various projective spaces. The triple system of collinear triples in projective space is a 1-$(q,3)$-system by letting the q-sets be lines. Hence results about the independence number and chromatic number of 1-$(q,3)$-systems have connections to questions about large caps in projective spaces, which is a fundamental problem in finite geometry.

The chromatic number $\chi(H)$ of a hypergraph H is the minimum number of colors required to color the vertex set of H so that no edge of H is monochromatic. A fundamental question about hypergraphs, first systematically investigated in the seminal work of Erdős and Lovász [12], is to determine the maximum chromatic number of a hypergraph with a specified number of edges. In this paper, we consider this problem for ℓ-(q,k)-systems. Call a clique with q vertices a *q-clique*.

Definition 1. *Given integers $1 \le \ell < k \le q$ and $e \ge 1$, let $f_\ell(e,q,k)$ be the maximum chromatic number of an ℓ-(q,k)-system where the number of q-cliques is e.*

Perhaps the most natural and interesting case is $\ell = k-1$ so in this case we use the simpler terminology (q,k)*-system* and write $f(e,q,k) = f_{k-1}(e,q,k)$. We are interested in $f_\ell(e,q,k)$ when k is fixed and both q and e are large. Special cases of this function have been extensively studied in the past. For example, the celebrated Erdős-Faber-Lovász conjecture [11, 16, 17] for graphs, which states that the maximum chromatic number of a collection of q almost disjoint q-cliques is q, is the statement $f(q,q,2) = q$. Another example when $k > 2$ is fixed, is $f_\ell(e,k,k)$, the largest possible chromatic number of partial designs, a classical question first studied by Erdős and Lovász [12], and subsequently by Ajtai et.al. [1] whose results were sharpened by Rödl and Šiňajová [21] and many others.

It is well-known that for k fixed, every k-graph with m edges has chromatic number $O(m^{1/k})$. Indeed, color the vertices randomly and independently with $O(m^{1/k})$ colors. The expected number of monochromatic edges is $O(m^{1-(k-1)/k}) = O(m^{1/k})$. Now assign new colors one by one to some vertex inside each monochromatic edge to get a proper coloring. Altogether we used at most $O(m^{1/k})$ colors and there are no monochromatic edges.

An (e,q,k)-system is a (q,k)-system where the number of q-cliques is e. An (e,q,k)-system has exactly $e\binom{q}{k}$ edges, so for fixed k, the argument above yields

$$f(e,q,k) = O(e^{1/k}q).$$

If there is an n-vertex q-graph H such that every k-set of vertices lies in exactly one edge, then the chromatic number of the (q,k)-system comprising the q-cliques of H is exactly $n/(k-1)$. The number of q-sets in H is $e = \binom{n}{k}/\binom{q}{k} = (n)_k/(q)_k$ and hence the chromatic number of this (q,k)-system is of order $e^{1/k}q$. This shows that $f(e,q,k) = \Theta(e^{1/k}q)$ for $e = (n)_k/(q)_k$. These designs are generally known to exist only when the number of cliques is exponential in the clique size – see Glock, Kühn, Loh and Osthus [13] and Keevash [18], and even near designs are generally known only to exist when the number of cliques is exponential in the clique size – see Rödl [20]. In this paper, we construct near designs where the number of cliques is polynomial in the clique size, and show that they have large chromatic number.

Our first result below is a similar lower bound (construction) for $f(e, q, k)$ when e is polynomial in q, which we prove in Section 2. The construction combines algebraic and probabilistic ideas, by taking a random restriction of $(q, 2)$-systems obtained from an affine plane – see also [2].

Theorem 2. *Fix $k \geq 2$. Suppose that $q > k$ is sufficiently large and $2^q > e > (50q)^k$. Then there exists an (e, q, k)-system with chromatic number at least $\Omega(e^{1/k}q)$. Consequently, for this choice of parameters, $f(e, q, k) = \Theta(e^{1/k}q)$.*

The case $k = 2$ deserves further mention. Hindman [14] was the first to observe that the determination of $f(q, q, 2)$ is equivalent to determining the edge chromatic number of the dual hypergraph, and using this formulation Chang and Lawler [4] gave the following nontrivial upper bound:

Proposition 3. ([4]) *Suppose that $2 < q < e < q^2$. Then $e/4 < f(e, q, 2) < 3e/2$.*

The lower bound in the proposition is proved as follows: let p be a prime such that $e^{1/2}/2 < p \leq e^{1/2}$ and let $A(2, p)$ be the affine plane of order p. Form H by enlarging each line of $A(2, p)$ by adding $q - p \geq q - e^{1/2} \geq 0$ new vertices such that distinct lines have disjoint enlargements. The resulting $(q, 2)$-system H has $p^2 \leq e$ q-cliques, and we may add disjoint q-sets arbitrarily so that we have exactly e cliques. In a proper coloring, every two vertices in $A(2, p)$ must receive distinct colors, and hence $\chi(H) \geq p^2 > e/4$.

If we use the prime number theorem, then as $q \to \infty$, this construction yields $f(e, q, 2) \geq (1 + o(1))e$. Kahn [16] proved an upper bound for $f(e, q, 2)$ that is asymptotically optimal in the range $e < q^2$ so as $q \to \infty$,

$$f(e, q, 2) = (1 + (1))e \qquad \text{for} \qquad q \leq e \leq q^2.$$

There is a further improvement $f(e, q, 2) \leq e$ for e sufficiently large [17].

The next case $k = 3$ seems wide open in the case that e is small. For $e > q^3$, we have $f(e, q, 3) = \Theta(e^{1/3}q)$ by Theorem 2. For $q^{3/2} < e < q^3$, we have the lower bound $\Omega(e^{2/3})$ using circles in an inversive plane – see Dembowski [8] for background on inversive planes. We take the circles in an inversive plane of order $(1 + o(1))e^{1/3}$ and adding disjoint sets of new points to the circles to create q-sets. Any three points in the inversive plane lie in a circle and hence an edge of our hypergraph, so the independence number is at most two, and the chromatic number is at least half the number of vertices in the inversive plane which is $\Theta(e^{2/3})$. Apart from this we have no nontrivial lower or upper bounds in the range $q < e < q^3$.

Problem 4. *Determine the order of magnitude of $f(e, q, 3)$ in the range $q < e < q^3$.*

Another interesting case is $f_1(e,q,k)$ when $k>2$ (the q-sets form a linear q-graph). Here we prove the following theorem which gives bounds that get closer as k increases. The lower bound is obtained via a random greedy algorithm. It is one of the few instances where a constrained random q-graph process on n vertices is analyzed where q is polynomial in n.

Theorem 5. *Let $k,e \geq 3$ and $q \geq 1$. There exist positive constants c, C such that*

$$c\left(\frac{e^{1/2}}{\log e}\right)^{\frac{1}{k-1}} q^{1-\frac{2}{k-1}} < f_1(e,q,k) < C\left(\frac{e^{1/2}}{\log e}\right)^{\frac{1}{k-1}} q$$

where the lower bound holds for $e > q^{2k+3}$.

We prove the upper bound in Theorem 5 in Section 3 and the lower bound in Section 4. If q is fixed and $e \to \infty$, then this theorem gives

$$f_1(e,q,k) = \Theta\left(\frac{e^{1/2}}{\log e}\right)^{\frac{1}{k-1}}$$

thereby determining the order of magnitude of $f_1(e,q,k)$ as $e \to \infty$. We believe that the upper bound in Theorem 5 is sharp in order of magnitude for q sufficiently large and e at least a polynomial function of q.

Conjecture 6. *Fix $k \geq 3$. There exists $C = C_k > 0$ such that if $e > q^C$, then*

$$f_1(e,q,k) = \Theta\left(\left(\frac{e^{1/2}}{\log e}\right)^{\frac{1}{k-1}} q\right) \qquad (q \to \infty).$$

In a forthcoming paper we will prove the conjecture up to a polylogarithmic factor in e. It is an interesting wide open problem to study the behavior of $f_1(e,q,k)$ for smaller values of e. The smallest case $k=3$ naturally gives rise to a particular class of 1-$(e,q,3)$ systems, namely the collection of collinear triples in lines in finite projective or affine planes. In fact, one possible way to obtain 1-$(q,3)$ systems is to take projective planes with few caps, and then take a random restriction of the points. This suggests another fundamental question, namely to count the number of caps in a finite projective/affine plane.

Define I_q to be the maximum number of caps in any projective plane of order q. In a Desarguesian plane the normal rational curve yields a cap of size at least $q+1$ and since all subsets of a cap are caps, we have at least 2^{q+1} caps. However, in general projective planes, the largest known caps that are guaranteed to exist are of order $(q\log q)^{1/2}$ giving $I_q > 2^{c\,(q\log q)^{1/2}}$. We give an improvement of this lower bound:

Theorem 7. *There is an absolute constant c such that for all sufficiently large q for which I_q is defined,*

$$I_q > 2^{c\,q^{1/2}(\log q)^{3/2}}.$$

We prove Theorem 7 in Section 5. It is a challenging open problem to construct projective planes of order q where the largest caps have size substantially less than $q+1$ – see

Hirschfeld and Storme [15] for a survey on the sizes of caps in planes. For the known projective planes and affine planes there are at least 2^{q+1} caps, and this is tight up to a factor $2^{o(q)}$ due to the results of Bhowmick and Roche-Newton [3] and Liu, Nie and Zeng [9]. If the bound in Theorem 7 is tight for some projective plane P, then the hypergraph of collinear triples in such a plane is a 1-$(e, q, 3)$-system with $e = q^2 + q + 1$ and independence number at most $cq^{1/2}(\log q)^{3/2}$, and therefore chromatic number at least $\Omega(e^{1/4}/(\log e)^{3/2} \cdot q)$, and this would be an explicit construction giving $f_1(e, q, 3) = \Omega(e^{1/4}/(\log e)^{3/2} \cdot q)$. It may be that there is a projective plane of order q whose largest cap has size of order $(q \log q)^{1/2}$, which would determine the order of magnitude of $f_1(e, q, 3)$ to be $e^{1/4}/(\log e)^{1/2} \cdot q$.

2 Random restrictions of affine planes

We will use the following version of the Chernoff bounds.

Lemma 8 (Multiplicative Chernoff Bounds). *Suppose $X_1, \ldots, X_n$ are independent random variables taking values in $\{0, 1\}$. Let X denote their sum and let $\mu = E[X]$ denote the expected value of X. Then for any $\delta > 0$,*

$$\Pr(X \leq (1 - \delta)\mu) < \exp(-\delta^2\mu/2)$$

$$\Pr(X \geq (1 + \delta)\mu) < \exp(-\delta^2\mu/(2 + \delta)).$$

Proof of Theorem 2. By the prime number theorem, there is a prime number $Q \geq q$ such that $(1/2)e < Q^k + Q \leq e$. Let $V = \mathbb{F}_Q \times \mathbb{F}_Q$. Given a polynomial $p(x)$ over $\mathbb{F}_Q$ let $S(p(x)) = \{p(x) : x \in \mathbb{F}_Q\}$. Let H_Q be the Q-graph with vertex set V and edge set

$$\{S(p(x)) : \deg(p(x)) < k)\} \cup \{C_x : x \in \mathbb{F}_Q\}$$

where $C_x = \{(x, y) : y \in \mathbb{F}_Q\}$ is the column of x. The number of edges in H_Q is $Q^k + Q \leq e$.

Let W be a random subset of V obtained by picking each element of V independently with probability $p = \frac{q}{10Q}$. Given an edge $f \in H_Q$, the expected size of $f \cap W$ is $q/10$ and the multiplicative Chernoff bound with $m = q/10$ and $\delta = 9$ implies that the probability that $|f \cap W| > q$ is at most $\exp(-81q/110)$. The number of edges f in H_Q is $Q^k + Q \leq e < 2^q$, so the union bound implies that

$$\Pr(\exists f : |f \cap W| > q) < 2^q \exp(-81q/110) < (0.96)^q.$$

For C a sufficiently large constant, the multiplicative Chernoff bound (with $m = pQ^2$ and $\delta = 1 - 1/C$) yields

$$\Pr(|W| < pQ^2/C) \leq \exp(-0.49pQ^2) = \exp(-0.049qQ).$$

Since $e > (50q)^k$, we have $Q > (e/4)^{1/k} \geq 25q$, and $\exp(-0.049qQ) < \exp(-q^2)$. Since $(0.96)^q + \exp(-q^2) < 1$, there is a set W such that $|f \cap W| \leq q$ for all f and $|W| = \Omega(qQ)$. Let H' be the k-graph with vertex set W whose edges are the $Q^k + Q$ cliques $\binom{f \cap W}{k}$.

We now prove that $\alpha(H') \le (k-1)^2$.

Claim. Given $(x_1, y_1), \ldots, (x_k, y_k)$ in $\mathbb{F}_Q \times \mathbb{F}_Q$ with the x_is distinct, there is a (unique) polynomial $p(x)$ of degree less than k with $p(x_i) = y_i$ for $i \in [k]$.

Proof of Claim. Write $p(x) = \sum_{j=0}^{k-1} a_j x^j$. Then the conclusion of the Claim is equivalent to the matrix equation $Ba = y$, where B is the k by k Vandermonde matrix with parameters $x_1, \ldots, x_k$, $a = (a_0, \ldots, a_{k-1})^T$ and $y = (y_1, \ldots, y_k)^T$. Since the x_is are distinct, B is invertible and hence there is a unique solution a. ∎

Pick a set $I \subset W$ of vertices of size at least $(k-1)^2+1$. If I has at least k vertices in some column C_x, then these k vertices lie in $C_x \cap W$ and hence lie in an edge of H'. Therefore, by the pigeonhole principle, I has at least k vertices in distinct columns. By the Claim, these k vertices lie in a unique $S(p(x))$ and hence lie in an edge of H'. This proves that $\alpha(H') \le (k-1)^2$.

We now modify H' by adding $q - |f \cap W|$ new vertices to each set $f \cap W$ to make a clique of size exactly q (these new sets of vertices are pairwise disjoint). This produces an (e, q, k)-system with chromatic number at least $|W|/(k-1)^2 = \Omega(qQ) = \Omega(e^{1/k}q)$. ∎

3 The upper bound in Theorem 5

Here we prove the upper bound in Theorem 5. We will need the following result.

Theorem 9 ([6])**.** *Fix $k \ge 3$. Let H be a k-graph with maximum degree at most d such that for each $s \in \{2, \ldots, k-1\}$, the maximum number of edges containing an s-set of vertices is $O(d^{(k-s)/(k-1)}/f)$. Then the chromatic number of H is $O((d/\log f)^{1/(k-1)})$.*

Note that the number of k-sets in a (q, k)-system where the number of q-cliques is e is $e\binom{q}{k}$ so the trivial bound is $f_1(e, q, k) = O(e^{1/k}q)$.

Theorem 10. *Fix $k \ge 3$. For $e > q$, $f_1(e, q, k) = O(q(e^{1/2}/\log e)^{1/(k-1)})$ as $q \to \infty$.*

Proof. Let H be a 1-(q, k)-system with e edges. Put $d := e^{1/2}q^{k-1}$. Let $A = \{v \in V(H) : d(v) \le d\}$ and $B = V(H) \setminus B$, and denote by $\Delta(H)$ the maximum degree of H. By definition, $\Delta(H[A]) \le d$. For each s-set S in A with $2 \le s \le k-1$, the number of edges in $H[A]$ containing S is at most $\binom{q-s}{k-s} < q^{k-s} \le d^{(k-s)/(k-1)}/e^{\epsilon}$ for $\epsilon = (k-s)/(2k-2)$. Hence Theorem 9 with $f = e^{\epsilon}$ implies that there is a proper coloring of $H[A]$ with at most $O((d/\log e)^{1/(k-1)}) = O((e^{1/2}/\log e)^{1/(k-1)} \cdot q)$ colors. Since

$$ke\binom{q}{k} \ge \sum_{v \in B} d(v) \ge |B|d = |B|e^{1/2}q^{k-1}$$

we obtain $|B| = O(e^{1/2}q)$. Now consider $v \in B$. The edges in H containing v that lie within B are all in subsets of q-cliques containing v. Let $A_1, \ldots, A_p$ be the set of q-cliques containing v that have at least k vertices in B and let $a_i = |A_i \cap B|$. Then the degree of

v in $H[B]$ is $\sum_i \binom{a_i-1}{k-1}$. Since $A_i \cap A_j = \{v\}$, we have $\sum(a_i - 1) < |B| = O(e^{1/2}q)$. The quantity $\sum_i \binom{a_i-1}{k-1}$ subject to this constraint is maximized when as many of the a_i are as large as possible and the rest are as small as possible. Since $a_i \le q$, we obtain

$$\sum_i \binom{a_i - 1}{k-1} \le \frac{|B|-1}{q-1}\binom{q-1}{k-1} = O(e^{1/2}q^{k-1}) = O(d).$$

Hence the maximum degree of $H[B]$ is $O(d)$. As we argued within $H[A]$, for $2 \le s \le k-1$, each s-set in B lies in at most $d^{(k-s)/(k-1)}/e^{\epsilon}$ edges of $H[B]$, so we can properly color $H[B]$ with $O((e^{1/2}/\log e)^{1/(k-1)} \cdot q)$ colors. We always use colors that have not been used in $H[A]$. In particular, this implies that if there is a k-set that has vertices in both A and B, then it will not be monochromatic in our coloring. The resulting coloring is a proper coloring of H with $O((e^{1/2}/\log e)^{1/(k-1)} \cdot q)$ colors. ∎

4 The lower bound for Theorem 5

In this section we prove the lower bound in Theorem 5 using a randomized greedy algorithm: $f(e, q, k) = \Omega((e^{1/2}/\log e))^{1/(k-1)} \cdot q^{1-2/(k-1)})$ for $e > q^{2k+3}$.

Fix $k \ge 3$. Consider the random greedy (q, k)-process on $n = \lfloor (q-1)\sqrt{e} \rfloor$ points: We pick a q-set e_1 of $[n]$ at random. Given that we have picked $e_1, \ldots, e_i$, we pick a q-set e_{i+1} randomly (with equal probability) from all other q-sets that do not intersect any of $e_1, \ldots, e_i$ in more than one point. Eventually we obtain a (random) q-graph G^q with e edges, and also a random k-graph $H = H^k = \cup_{i=1}^{e} \binom{e_i}{k}$, which has $e\binom{q}{k}$ edges.

We write $f \gg g$ to mean that there is a (large) positive constant $c = c_k$ such that $f \ge cg$. From now on, we select t such that

$$n \gg tq^2 \qquad \text{and} \qquad t \gg 10k\left(\frac{n^{k-2}\log n}{q^{k-4}}\right)^{\frac{1}{k-1}}.$$

A short calculation shows this is possible, since $e > q^{2k+3}$ and $n = \lfloor (q-1)\sqrt{e} \rfloor$.

Fix a t-set I of $[n]$ and let us calculate the probability that I is an independent set in H^k. For $i \ge 0$, let

$$W_j = \left\{ S \in \binom{[n]}{q} : |S \cap I| = j \right\}$$

and put $M = \cup_{j=k}^{q} W_j$ and $m := |M|$. Here M stands for *missing* q-sets since these q-sets cannot be present in G^q as I is an independent set in H^k. Note that $w_j := |W_j| \le \binom{t}{j}\binom{n}{q-j}$. Say that $A \in W_j$ *blocks* $B \in M$ if $|A \cap B| \ge 2$ and let b_j be the number of sets in M blocked by an $A \in W_j$ (it is the same for all A).

We then have

$$b_0 < q^2 t^k \binom{n}{q-k-2}$$

$$b_1 \le q\binom{t}{k-1}\binom{n}{q-k-1} + q^2 t^k \binom{n}{q-k-2} < qt^{k-1}\binom{n}{q-k-1}$$

and for $2 \le j \le k-1$,

$$b_j \le \binom{j}{2}\binom{t}{k-2}\binom{n}{q-k} + jqt^{k-1}\binom{n}{q-k-1} + q^2t^k\binom{n}{q-k-2} < 2t^{k-2}\binom{n}{q-k},$$

where we use $n/tq^2 \to \infty$ in the last two displays. Note that $m = \Theta(t^k\binom{n}{q-k})$. Let $a_j = m/b_j$ so that

$$a_0 = \Theta\left(\frac{n^2}{q^4}\right) \qquad a_1 = \Theta\left(\frac{tn}{q^2}\right) \qquad a_j = \Theta(t^2) \qquad \text{for } 2 \le j \le k-1.$$

Claim. If I is an independent set, then at least a_j/k edges of W_j must be present in G^q for some $0 \le j \le k-1$.

Proof. If not, since no set of M is in G^q, the number of sets in M that are blocked is less than $\sum_{j=0}^{k-1} b_j(a_j/k) \le m$. This means that some edge of M would be in G^q, contradicting the fact that I is an independent set.

Let $G_i = G_i^q$ be the (random) q-graph obtained after i edges have been added and let e_i be the q-set added at step i. Define

$$\ell := \min\{i : |W_j \cap G_i| \ge a_j/3k \text{ for some } j = 0, 1, \ldots, k-1\}.$$

In words, ℓ is the smallest index such that G_i contains at least $a_j/3k$ edges from W_j for some j.

For $0 \le j \le k-1$, let A_j be the event that I is an independent set and $e_\ell \in W_j$. The Claim, the definition of ℓ, and the union bound imply that

$$\Pr(I \text{ is independent}) \le \sum_{j=0}^{k-1} \Pr(A_j).$$

Let $m_i = |M \setminus G_i|$. Since $|W_j \cap G_\ell| \le a_j/3k$ for each $j \in \{0, 1, \ldots, k-1\}$, we have for $i \le \ell$,

$$m_i \ge m_\ell \ge m - \sum_{j=0}^{k-1} b_j(a_j/3k) \ge m - km/3k > m/2.$$

For each $S \subset \{1, \ldots, e\}$, define the event

$$A_0(S) := \{G^q \in A_0 : e_i \in W_0 \Leftrightarrow i \in S\}.$$

These events are disjoint for distinct S and hence

$$\Pr(A_0) = \sum_{S \subset [e]} \Pr(A_0(S)).$$

If $\Pr(A_0(S)) > 0$, then since I is an independent set there is a subset $S' \subset S$ with $|S'| \ge a_0/3k$ and $m_i \ge m/2$ for all $i \in S'$. Hence we can further write

$$\Pr(A_0) = \sum_{|S| \ge a_0/3k} \Pr(A_0(S)).$$

Write $w_{j,i} = |W_j \setminus G_i|$ and $r_i = w_{1,i} + \cdots + w_{k-1,i}$ for $j = 0, 1, \ldots, k-1$. Then

$$\begin{aligned}
\Pr(A_0) &\le \sum_{|S| \ge \frac{a_0}{3k}} \left(\prod_{i \in S} \Pr(e_i \in W_0) \prod_{i \notin S} \Pr(e_i \in W_1 \cup \cdots \cup W_{k-1}) \right) \\
&= \sum_{|S| \ge \frac{a_0}{3k}} \prod_{i \in S} \left(\frac{w_{0,i}}{w_{0,i} + m_i + r_i} \right) \prod_{i \notin S} \left(\frac{r_i}{w_{0,i} + m_i + r_i} \right) \\
&= \sum_{|S| \ge \frac{a_0}{3k}} \prod_{i \in S} \left(\frac{w_{0,i}}{w_{0,i} + m_i} \right) \left(\frac{w_{0,i} + m_i}{w_{0,i} + m_i + r_i} \right) \prod_{i \notin S} \left(\frac{r_i}{w_{0,i} + m_i + r_i} \right) \\
&= \sum_{|S| \ge \frac{a_0}{3k}} \prod_{i \in S} \left(1 - \frac{m_i}{w_{0,i} + m_i} \right) \left(\frac{w_{0,i} + m_i}{w_{0,i} + m_i + r_i} \right) \prod_{i \notin S} \left(\frac{r_i}{w_{0,i} + m_i + r_i} \right) \\
&\le \exp\left(-\frac{m a_0}{6k w_0} \right) \sum_{|S| \ge \frac{a_0}{3k}} \prod_{i \in S} \left(\frac{w_{0,i} + m_i}{w_{0,i} + m_i + r_i} \right) \prod_{i \notin S} \left(\frac{r_i}{w_{0,i} + m_i + r_i} \right) \\
&\le \exp\left(-\frac{m a_0}{6k w_0} \right) \sum_{S \subset [e]} \prod_{i \in S} \left(\frac{w_{0,i} + m_i}{w_{0,i} + m_i + r_i} \right) \prod_{i \notin S} \left(\frac{r_i}{w_{0,i} + m_i + r_i} \right) \\
&= \exp\left(-\frac{m a_0}{6k w_0} \right) \prod_{i=1}^{e} \left(\frac{w_{0,i} + m_i}{w_{0,i} + m_i + r_i} + \frac{r_i}{w_{0,i} + m_i + r_i} \right) \\
&= \exp\left(-\frac{m a_0}{6k w_0} \right).
\end{aligned}$$

We observe that $\binom{n}{t} \Pr(A_0) < 1/k$ since this follows from

$$\frac{m a_0}{6k w_0} = \frac{m^2}{6k b_0 w_0} > \frac{[\binom{t}{k}\binom{n}{q-k}]^2}{6kq^2 t^k \binom{n}{q-k-2}\binom{n}{q}} > t \log n$$

which holds due to $t^{k-1} \gg 10kn^{k-2} \log n / q^{k-4}$. Similarly, $\binom{n}{t} \Pr(A_1) < 1/k$ follows from

$$\frac{m^2}{6k b_1 w_1} > \frac{[\binom{t}{k}\binom{n}{q-k}]^2}{6kq t^{k-1} \binom{n}{q-k-1} t \binom{n}{q-1}} \gg t \log n$$

using the weaker bound $t^{k-1} \gg n^{k-2} \log n / q^{k-3}$. Finally, $\binom{n}{t} \Pr(A_j) < 1/j$ for each $2 \le j \le k-1$ follows from

$$\frac{m^2}{6k b_j w_j} > \frac{[\binom{t}{k}\binom{n}{q-k}]^2}{12k t^{k-2+j} \binom{n}{q-k}\binom{n}{q-j}} \gg t \log n.$$

This is equivalent to

$$t > \left(\frac{n^{k-j} \log n}{q^{k-j}} \right)^{\frac{1}{k+1-j}}.$$

Using $t \gg 10k(n^{k-2} \log n / q^{k-4})^{1/(k-1)}$ this follows from

$$\left(\frac{n^{k-2} \log n}{q^{k-4}} \right)^{k+1-j} \gg \left(\frac{n^{k-j} \log n}{q^{k-j}} \right)^{k-1}$$

which is equivalent to $n^{j-2} \gg (\log n)^{j-2} q^{3j-2k-4}$ and this is trivial using $n > q$ and $j < k$.

So with positive probability, the independence number of H^k is less than t.

Certainly $|E(G^q)| \leq \binom{n}{2}/\binom{q}{2}$ since G^q is a (q,k)-system. Recalling $n = \lfloor (q-1)\sqrt{e} \rfloor$, we have $|E(G^q)| \leq e$, and we may assume by adding disjoint edges if needed that $|E(G^q)| = e$. The chromatic of H^k is at least

$$\frac{n}{t} = \Omega\left(\left(\frac{nq^{k-4}}{\log n}\right)^{\frac{1}{k-1}}\right) = \Omega\left(\frac{e^{1/2}}{\log e}\right)^{\frac{1}{k-1}} \cdot q^{1-\frac{2}{k-1}}.$$

Consequently,

$$f_1(e,q,k) = \Omega\left(\frac{e^{1/2}}{\log e}\right)^{\frac{1}{k-1}} \cdot q^{1-\frac{2}{k-1}}$$

which completes the proof of Theorem 5. ∎

5 Counting caps in affine/projective spaces

In this section we prove Theorem 7. We require the following lemma from [6]. Recall that $\tau(F)$ is the minimum size of a vertex subset of F that intersects every edge of F.

Lemma 11. *Suppose F is an s-uniform hypergraph, and z_i, $i \in V(F)$ are independent random indicator variables with* $\Pr[z_i = 1] = p$*, for all $i \in V(F)$. Let*

$$F' = \{A \in F : \forall i \in A, z_i = 1\}.$$

Suppose there exists $\alpha > 0$ such that $|F|p^{s(1-\alpha)} < 1$. Then for any $c \geq e2^s s\alpha$,

$$\Pr[\tau(F') > s^2(c/\alpha)^{s+1}] \leq s^2|V(F)|^{s-1}p^c.$$

The reason we need to use the lemma above is to take care of codegrees that may be of logarithmic size (the codegree of a pair of vertices is the number of edges containing them both). We also need the following result from [7]; the result as stated in [7] applies to linear hypergraphs but standard methods imply the same bound for hypergraphs with bounded maximum codegree.

Theorem 12 ([7], Theorem 3)**.** *Fix $s > 0$. There exists a constant $c = c_s > 0$ such that the number of independent sets in every n-vertex 3-graph with average degree d and maximum codegree s is at least $2^{c\,n(\log d)^{3/2}/d^{1/2}}$.*

Proof of Theorem 7. A cherry is the 5 vertex 3-graph comprising three edges, every two of which share the same two vertices. Let H be a $(q,3)$-system with $n = q^2$ vertices, average degree at most q^3, and maximum codegree at most q. Note that the collinear triples of any affine plane of order q can be viewed as such a $(q,3)$-system H.

Our plan is to take a random induced subgraph of H on $q^{3/5-\epsilon}$ vertices where there is a small set of vertices that touches every cherry. To this end we apply Lemma 11 with F

being the 5-graph of copies of cherries in H and $p = q^{-7/5-\epsilon}$. Letting α be sufficiently small in terms of ϵ, we obtain

$$|F|p^{(1-\alpha)s} < n^2 q^3 p^{(1-\alpha)s} < q^7 q^{-5(1-\alpha)(7/5+\epsilon)} = o(1).$$

Hence for c a large constant, Lemma 11 and standard Chernoff bounds yield that with probability greater than 0.9 say, a random induced subgraph H' of H with p as above has $m = \Theta(q^{3/5-\epsilon})$ vertices, average degree $d = O(p^2q^3) = O(q^{1/5-2\epsilon})$ and has a set S of $O(1)$ vertices whose removal makes it F-free. Since $H' - S$ has no cherry, it has maximum codegree at most two. Theorem 12 now implies that the number of independent sets in $H' - S$, and hence also in H, is at least $2^{c\, q^{1/2}(\log q)^{3/2}}$. ∎

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